

Figure 7.16. Chemical potential of a noninteracting, nonrelativistic Fermi gas in a three-dimensional box, calculated numerically as described in Problem 7.32. At low temperatures μ is given approximately by equation 7.66, while at high temperatures μ becomes negative and approaches the form for an ordinary gas obeying Boltzmann statistics.

Now admittedly, that was a lot of work just to get a factor of $\pi^2/4$ (since we had already guessed the rest by dimensional analysis). But I've presented this calculation in detail not so much because the *answer* is important, as because the *methods* are so typical of what professional physicists (and many other scientists and engineers) often do. Very few real-world problems can be solved exactly, so it's crucial for a scientist to learn when and how to make approximations. And more often than not, it's only *after* doing the hard calculation that one develops enough intuition to see how to guess most of the answer.

Problem 7.29. Carry out the Sommerfeld expansion for the energy integral (7.54), to obtain equation 7.67. Then plug in the expansion for μ to obtain the final answer, equation 7.68.

Problem 7.30. The Sommerfeld expansion is an expansion in powers of kT/ϵ_F , which is assumed to be small. In this section I kept all terms through order $(kT/\epsilon_F)^2$, omitting higher-order terms. Show at each relevant step that the term proportional to T^3 is zero, so that the next nonvanishing terms in the expansions for μ and U are proportional to T^4 . (If you enjoy such things, you might try evaluating the T^4 terms, possibly with the aid of a computer algebra program.)

Problem 7.31. In Problem 7.28 you found the density of states and the chemical potential for a two-dimensional Fermi gas. Calculate the heat capacity of this gas in the limit $kT \ll \epsilon_F$. Also show that the heat capacity has the expected behavior when $kT \gg \epsilon_F$. Sketch the heat capacity as a function of temperature.

Problem 7.32. Although the integrals (7.53 and 7.54) for N and U cannot be carried out analytically for all T , it's not difficult to evaluate them numerically using a computer. This calculation has little relevance for electrons in metals (for which the limit $kT \ll \epsilon_F$ is always sufficient), but it is needed for liquid ${}^3\text{He}$ and for astrophysical systems like the electrons at the center of the sun.

- (a) As a warm-up exercise, evaluate the N integral (7.53) for the case $kT = \epsilon_F$ and $\mu = 0$, and check that your answer is consistent with the graph shown

above. (Hint: As always when solving a problem on a computer, it's best to first put everything in terms of dimensionless variables. So let $t = kT/\epsilon_F$, $c = \mu/\epsilon_F$, and $x = \epsilon/\epsilon_F$. Rewrite everything in terms of these variables, and *then* put it on the computer.)

- (b) The next step is to vary μ , holding T fixed, until the integral works out to the desired value, N . Do this for values of kT/ϵ_F ranging from 0.1 up to 2, and plot the results to reproduce Figure 7.16. (It's probably not a good idea to try to use numerical methods when kT/ϵ_F is much smaller than 0.1, since you can start getting overflow errors from exponentiating large numbers. But this is the region where we've already solved the problem analytically.)
- (c) Plug your calculated values of μ into the energy integral (7.54), and evaluate that integral numerically to obtain the energy as a function of temperature for kT up to $2\epsilon_F$. Plot the results, and evaluate the slope to obtain the heat capacity. Check that the heat capacity has the expected behavior at both low and high temperatures.

Problem 7.33. When the attractive forces of the ions in a crystal are taken into account, the allowed electron energies are no longer given by the simple formula 7.36; instead, the allowed energies are grouped into **bands**, separated by **gaps** where there are no allowed energies. In a **conductor** the Fermi energy lies within one of the bands; in this section we have treated the electrons in this band as “free” particles confined to a fixed volume. In an **insulator**, on the other hand, the Fermi energy lies within a gap, so that at $T = 0$ the band below the gap is completely occupied while the band above the gap is unoccupied. Because there are no empty states close in energy to those that are occupied, the electrons are “stuck in place” and the material does not conduct electricity. A **semiconductor** is an insulator in which the gap is narrow enough for a few electrons to jump across it at room temperature. Figure 7.17 shows the density of states in the vicinity of the Fermi energy for an idealized semiconductor, and defines some terminology and notation to be used in this problem.

- (a) As a first approximation, let us model the density of states near the bottom of the conduction band using the same function as for a free Fermi gas, with an appropriate zero-point: $g(\epsilon) = g_0\sqrt{\epsilon - \epsilon_c}$, where g_0 is the same constant as in equation 7.51. Let us also model the density of states near the top

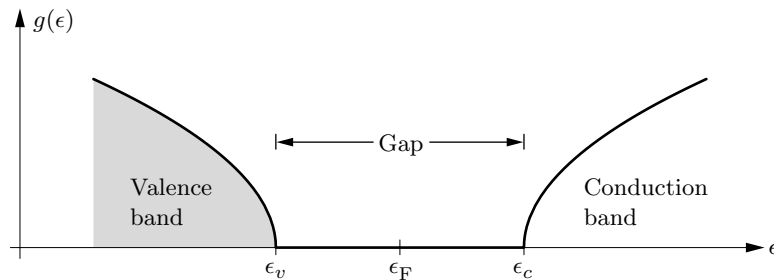


Figure 7.17. The periodic potential of a crystal lattice results in a density-of-states function consisting of “bands” (with many states) and “gaps” (with no states). For an insulator or a semiconductor, the Fermi energy lies in the middle of a gap so that at $T = 0$, the “valence band” is completely full while the “conduction band” is completely empty.