

17. Ladder Operators

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Operators in quantum mechanics aren't merely a convenient way to keep track of eigenvalues (measurement outcomes) and eigenvectors (definite-value states). We can also use them to streamline calculations, stripping away unneeded calculus (or explicit matrix manipulations) and focusing on the essential algebra.

As an example, let's now go back to the one-dimensional simple harmonic oscillator, and use operator algebra to find the energy levels and associated eigenfunctions.

Recall that the harmonic oscillator Hamiltonian is

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega_c^2x^2, \quad (1)$$

where p is the momentum operator, $p = -i\hbar d/dx$ (in this lesson I'll use the symbol p only for the *operator*, never for a momentum *value*). As in Lesson 8, it's easiest to use natural units in which we set the particle mass m , the classical oscillation frequency ω_c , and \hbar all equal to 1. Then the Hamiltonian is simply

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2, \quad (2)$$

with $p = -id/dx$.

The basic trick, which I have no idea how to motivate, is to define two new operators that are linear combinations of x and p :

$$a_- = \frac{1}{\sqrt{2}}(x + ip), \quad a_+ = \frac{1}{\sqrt{2}}(x - ip). \quad (3)$$

These are called the *lowering* and *raising* operators, respectively, for reasons that will soon become apparent. Unlike x and p and all the other operators we've worked with so far, the lowering and raising operators are not Hermitian and do not represent any observable quantities. We're not especially concerned with their eigenvalues or eigenfunctions; instead we'll focus on how they mathematically convert one *energy* eigenfunction into another. (Note: Most authors use the notation a and a^\dagger instead of a_- and a_+ , but Griffiths uses the $-/+$ notation, and I too think it's more appropriate. In conventional units, by the way, we would insert factors of m , ω_c , and \hbar into these definitions as needed to make a_- and a_+ dimensionless. It's a good exercise to figure out exactly how to do this, especially if you're even a little uncomfortable with my use of natural units.)

The first thing to notice about a_- and a_+ is that if you multiply them together, you almost get the Hamiltonian:

$$\begin{aligned} a_-a_+ &= \frac{1}{2}(x^2 + p^2 - i(xp - px)) \\ &= H - \frac{i}{2}[x, p] \\ &= H + \frac{1}{2}, \end{aligned} \quad (4)$$

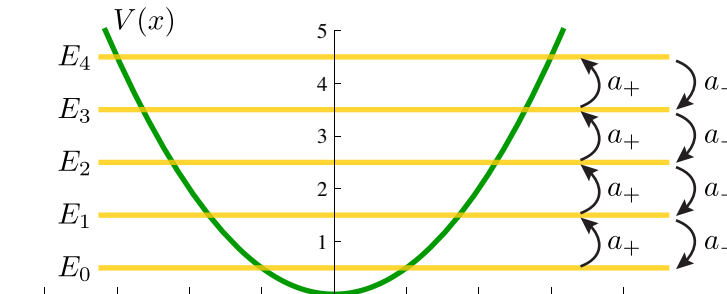
where I've used the basic commutator relation $[x, p] = i\hbar$ (with $\hbar = 1$) that you worked out for homework. Similarly, multiplying them together in the other order gives $a_+a_- = H - \frac{1}{2}$. We can therefore write the Hamiltonian as

$$H = a_-a_+ - \frac{1}{2} = a_+a_- + \frac{1}{2}. \quad (5)$$

Now suppose that ψ is any energy eigenfunction, so that $H\psi = E\psi$ for some E . Then I claim that if we act on ψ with either the raising or the lowering operator, we get *another* energy eigenfunction. To prove this, I need to check what happens when the Hamiltonian operator acts on this new function. Here's what happens in the case of $a_+\psi$:

$$\begin{aligned} H(a_+\psi) &= (a_+a_- + \frac{1}{2})(a_+\psi) \\ &= a_+(a_-a_+ + \frac{1}{2})\psi \\ &= a_+(H + 1)\psi \\ &= a_+(E + 1)\psi \\ &= (E + 1)(a_+\psi). \end{aligned} \quad (6)$$

So $a_+\psi$ is an energy eigenfunction, and its eigenvalue is exactly one unit greater than that of ψ itself. (In ordinary units, this one unit of energy would be $\hbar\omega_c$.) This is why a_+ is called the raising operator: it mathematically *raises* any energy eigenstate *up* the quantum ladder by one rung, to the next-highest energy eigenstate. There's no limit to how many times we can apply the raising operator, so this proves that a quantum harmonic oscillator has an infinite ladder of energy eigenstates, with equally spaced levels separated in energy by $\hbar\omega_c$.



By a completely analogous calculation you can show that a_- *lowers* any energy eigenstate ψ by one rung *down* the quantum ladder:

$$H(a_-\psi) = (E - 1)(a_-\psi). \quad (7)$$

This is why a_- is called the lowering operator. This result would seem to imply that our infinite ladder of energy eigenstates continues *downward* in energy without limit—but that can't possibly be the case, because there can't be any states with

negative energy when the potential energy is positive everywhere. The only way out is if, when a_- acts on the lowest-energy eigenstate (call it ψ_0), it gives *zero*:

$$a_- \psi_0 = 0. \quad (8)$$

Then equation 7 can still be true for ψ_0 even though there's no lower-energy state. And from this simple equation we immediately see that $H\psi_0 = \frac{1}{2}$, so the ground-state energy is half a unit, or $\frac{1}{2}\hbar\omega_c$. The remaining energies are integer steps above this, so we've proved that the energies of a quantum harmonic oscillator are as claimed in Lesson 8:

$$E_n = (n + \frac{1}{2})\hbar\omega_c, \quad n = 1, 2, 3, \dots \quad (9)$$

(If you're reading carefully you may have noticed a couple of loopholes in the logic I just described. How do we know the quantum ladder goes up indefinitely, rather than ending with a state ψ_{\max} for which $a_+\psi_{\max} = 0$? And how do we know that the ladder that starts with ψ_0 includes *all* the states? Could there be others that we've missed, perhaps lying in between the ones we've found? These loopholes are actually pretty easy to close, and I'd encourage you to think about them, but only after you're comfortable with the rest of the calculations and reasoning described here.)

To find the energy eigenfunctions that correspond to the eigenvalues E_n , we still need to use calculus—but at least the method is intuitive and straightforward. To find the ground-state eigenfunction we can use equation 8, which becomes an ordinary differential equation for $\psi_0(x)$ when we express the lowering operator in terms of the explicit momentum operator, $p = -i\hbar/dx$. The solution to the differential equation is $\psi_0(x) = Ae^{-x^2/2}$, just as we saw in Lesson 8 (and you can work out the normalization constant A if you like). Then, to find the first excited state, just apply the raising operator, also written in terms of $p = -i\hbar/dx$, to the ground state (and again work out the normalization constant if you need it). Keep applying the raising operator to work your way up the quantum ladder until the novelty wears off. As you might guess, it gets pretty tedious to work out more than the first few eigenfunctions by hand.

I hope you agree that the ladder-operator method is by far the most elegant way of solving the TISE for the simple harmonic oscillator. The bad news, though, is that no such elegant method exists for solving the TISE for other one-dimensional potential functions; the method worked here only because the Hamiltonian is quadratic in both p and x , allowing it to be factored, aside from an additive constant, into the product a_-a_+ . So why spend time learning a method—however elegant—whose applicability is so restricted? There are two reasons.

The first reason is that harmonic oscillators really are ubiquitous in nature, from vibrating molecules to elastic solids to the electromagnetic field and other fundamental fields of elementary particle physics. In continuous vibrating systems we refer to the units of excitation energy as *particles*, such as photons in the case of

the electromagnetic field, or as *quasiparticles*, such as phonons in the case of elastic vibrations. The study of these continuous vibrating quantum systems is called *quantum field theory*, and ladder operators are a fundamental tool of quantum field theorists. But we won't have time to explore quantum field theory in this course.

The second reason, though, is that ladder operators *will* come up again in this course in a somewhat different context: angular momentum. Instead of adding and removing energy, the ladder operators in that case will add and remove units of angular momentum along the z axis. They will therefore be an extremely useful tool in our study of systems with spherical symmetry, especially atoms.