Purcell Simplified: Magnetism, Radiation, and Relativity

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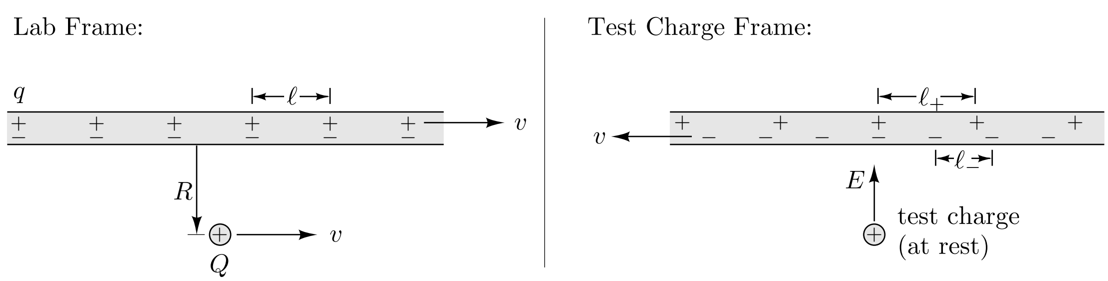
Dan Schroeder, Weber State University, http://physics.weber.edu/schroeder

Introductory Comments

- There's almost nothing original in this talk; Purcell gets all the credit.
- *Don't* use Purcell's book in an introductory course. (If you're tempted, read the reviews in Amazon.com.)
- I'm not presenting a complete curriculum; this material would occupy only 3–5 class sessions.
- I have prepared a 39-page set of typeset class notes, suitable for a calculus-based introductory course, which you can download from my web site.
- This material could also be adapted to an algebra-based course, with some loss of rigor.
- Prerequisites: 1. An understanding of electrostatic fields, including either Gauss's law or equivalent rules for field lines. 2. Familiarity with basic magnetic phenomena, e.g., parallel currents attract. 3. The basics of special relativity, including reference frames, length contraction, and the cosmic speed limit but *not* including the Lorentz transformation equations or relativistic dynamics.

Magnetism as a Consequence of Length Contraction

Model a current-carrying wire as a line of negative charges ($-q$) at rest and a line of positive charges ($+q$) moving to the right at speed v. The average linear separation between charges is ℓ. Consider a "test charge" Q moving parallel to the wire, at the same speed v (for simplicity). In the frame of the test charge it is at rest and so are the $+$ charges in the wire, but the $-$ charges are moving to the left. According to relativity, the distance between the $-$ charges is length-contracted to $\ell_- = \ell\sqrt{1-(v/c)^2}$, while the distance between the $+$ charges is un-length-contracted to $\ell_+ = \ell/\sqrt{1-(v/c)^2}$. Therefore the wire carries a net negative charge and exerts an attractive electrostatic force on the test charge. Back in the lab frame, we call this a magnetic force.



To calculate the strength of the force, first find the linear charge density of the wire in the test charge frame (assuming $v \ll c$ for simplicity):

$$\lambda = \frac{q}{\ell_+} - \frac{q}{\ell_-} = \frac{q}{\ell}\left(\sqrt{1-(v/c)^2} - \frac{1}{\sqrt{1-(v/c)^2}}\right) \approx \frac{q}{\ell}\left(1 - \frac{1}{2}\left(\frac{v}{c}\right)^2 - 1 - \frac{1}{2}\left(\frac{v}{c}\right)^2\right) = -\frac{q}{\ell}\left(\frac{v}{c}\right)^2. \tag{1}$$

Therefore the electrostatic force in this frame is

$$|\vec{F}_e| = Q|\vec{E}| = Q\cdot\frac{|\lambda|}{2\pi\epsilon_0 R} = \frac{Qqv^2}{2\pi\epsilon_0 R\ell c^2}, \tag{2}$$

where R is the distance of the test charge from the wire. The magnetic force in the lab frame has the same magnitude. Written in terms of the current, $I = qv/\ell$, the magnetic force is

$$|\vec{F}_m| = Qv\left(\frac{I}{2\pi\epsilon_0 c^2 R}\right). \tag{3}$$

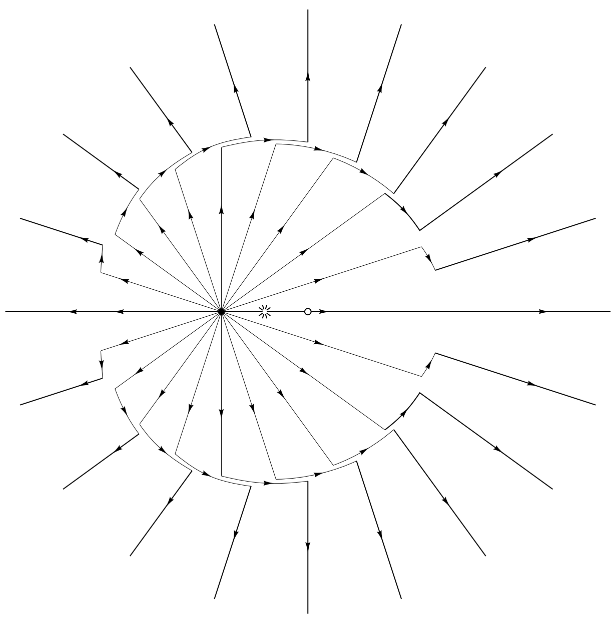
The expression in parentheses is what we call the magnetic field strength. Note that the quantity $1/(\epsilon_0 c^2)$ is what we usually call μ_0.

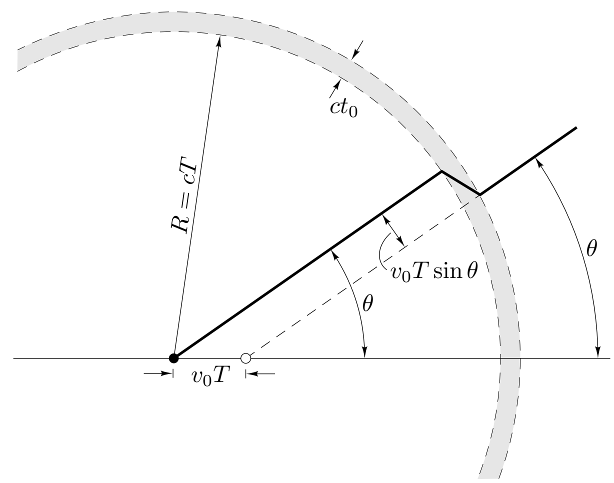
In a typical household wire, electrons move at a snail's pace: $v/c \sim 10^{-13}$, so the Lorentz factor differs from 1 by only about one part in 10^{26}. This tiny amount of length contraction is still observable, because the total charge of all the moving electrons is enough to exert enormous electrostatic forces (were it isolated).

The same derivation can be adapted to more complicated cases where the test charge has an arbitrary velocity, in either direction. To understand the case where the test charge is moving toward or away from the wire, you need to digress to show how the electric field of a point charge in motion is weaker in front of and behind the charge but stronger in the transverse directions. (This can be derived using length contraction and some simple gedanken experiments.)

Radiation as a Consequence of the Cosmic Speed Limit

Consider a charged particle moving at 1/4 the speed of light that bounces off a brick wall. Just after the bounce, the field nearby points directly away from the particle, but the field farther out points away from where the particle would be had it not bounced, because "electromagnetic news travels at the speed of light." Gauss's law (or continuity of field lines) requires that there be a large transverse electric field in the transition region between near and far, a spherical shell that expands at the speed of light. Students can learn to draw pictures of the electric field for any similar motion with instantaneous accelerations—even the periodic motion of a particle bouncing between two walls, which gives a crude picture of how a transmission antenna works.





To treat the radiation field quantitatively, consider the simpler example of a point charge q initially moving at speed $v_0 \ll c$ which then stops, decelerating uniformly for a duration of t_0. At a time $T \gg t_0$ after this happens, the pulse of radiation has reached a radius of $R = cT$. For a typical field line at an angle of θ, the geometry of the "kink" requires that the ratio of the transverse field to the radial field be

$$\frac{E_t}{E_r} = \frac{v_0 T \sin\theta}{ct_0} = \frac{aR\sin\theta}{c^2} \tag{4}$$

where a is the magnitude of the particle's acceleration. But the radial field is given by Coulomb's law, so the transverse field is

$$E_t = \frac{q}{4\pi\epsilon_0 c^2}\frac{a\sin\theta}{R}. \tag{5}$$

Notice that this falls off with distance as $1/R$, not $1/R^2$. The energy per unit volume stored in this field, proportional to $|\vec{E}|^2$, therefore falls of as $1/R^2$, so the total energy contained in the shell is unchanged as the shell expands. To calculate the power radiated you have to average over angles and also multiply by 2 to include the equal energy stored in the magnetic field. The result is the Larmor formula,

$$\text{Power radiated} = \frac{q^2 a^2}{6\pi\epsilon_0 c^3}, \tag{6}$$

a result that can be applied to understanding why the sky is blue and why a classical Rutherford atom is unstable.