Two Particles in One Dimension

I'll get right to the point:

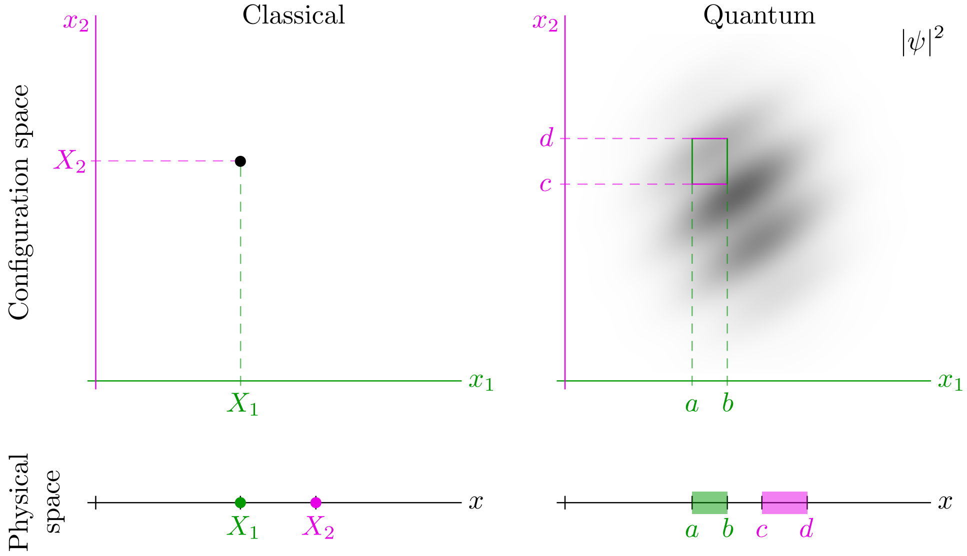
A system of two particles has only one wavefunction.

Please read that sentence aloud. Repeatedly. It takes some getting used to.

And it gets worse: A system of three particles, or four, or 10^{23}, also has only one wavefunction.

But let's start with just two particles, and say they're moving in just one dimension (to keep things as simple as possible). Then if x_1 represents a possible position of the first particle and x_2 represents a possible position of the second particle, the wavefunction is a function of two variables: $\psi(x_1, x_2)$. We say that ψ is a function of two-dimensional *configuration space*.

In classical mechanics, we can draw a *single* point in configuration space to indicate the positions of *both* particles. The illustration below (left) shows the case where particle 1 is at the particular location X_1, while particle 2 is at the particular location X_2. Notice that even though particle 2's position is plotted perpendicular to particle 1's in configuration space, its actual location in physical space is along the same one-dimensional line as particle 1's position.

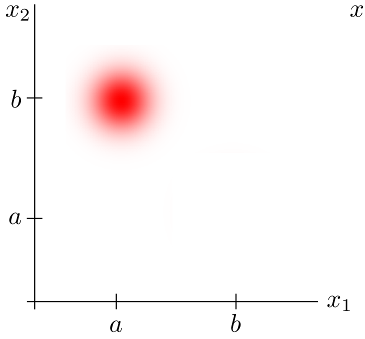


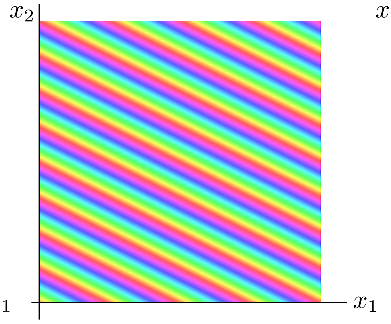
In quantum mechanics, particles generally don't have precise positions. But the square modulus of the wavefunction, $|\psi(x_1, x_2)|^2$, tells us the relative *probability* of finding particle 1 at x_1 and particle 2 at x_2. More precisely, if we integrate $|\psi(x_1, x_2)|^2$ over any desired ranges of x_1 and x_2, we get the probability of finding the two particles within those two ranges, respectively, if we were to measure *both* of their positions:

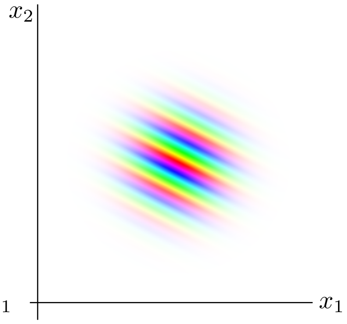
$$\begin{pmatrix} \text{Probability of finding} \\ a < x_1 < b \text{ and } c < x_2 < d \end{pmatrix} = \int_a^b dx_1 \int_c^d dx_2 \, |\psi(x_1, x_2)|^2. \tag{1}$$

The illustration above shows a particular choice for each of the intervals *ab* and *cd*. Remember that both intervals lie along the same line in physical space. Mathematically, a wavefunction for two particles in one dimension takes the same form as a wavefunction for one particle in two dimensions (x and y). It's just that the physical interpretation is trickier.

Here are some examples of wavefunctions for two particles in one dimension:





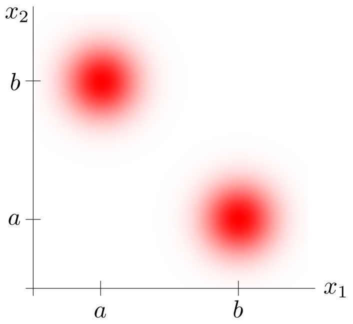


At left is a wavefunction for the situation in which both particles are fairly localized, with particle 1 near $x = a$ and particle 2 near $x = b$ (and with both particles having zero average momentum). The middle wavefunction shows a portion of a definite-momentum wavefunction, with both particles moving in the $+x$ direction but particle 2 having somewhat larger momentum than particle 1. At right is Gaussian wavepacket, or rather, a wavefunction for which both particles are in Gaussian wavepacket states. The formula for this wavefunction would have the form

$$\psi(x_1, x_2) = e^{-x_1^2}\, e^{ik_1x_1}\, e^{-x_2^2}\, e^{ik_2x_2}, \tag{2}$$

where for simplicity I've centered both particles' wavepackets at $x = 0$ and given them half-widths of about 1 unit. We could also imagine states in which one particle is much more localized than the other, and states in which one particle has much more momentum on average than the other, and so on.

Here is another possible two-particle wavefunction:



If this wavefunction described a single particle in two-dimensional space, we would simply call it a cat state, with two localized peaks that are well separated from each other along a diagonal direction; the particle would have a 50/50 chance of being found within either of the two regions. But we're actually talking here about two particles in one dimension! Particle 1 has a 50/50 chance of being found near either $x = a$ or $x = b$, and the same is true of particle 2. But *if* we find particle 1 near $x = a$, we're *guaranteed* to find particle 2 near $x = b$, and vice-versa. We can't possibly find both particles near a, nor can we find both particles near b. This wavefunction tells us that the two particles are in two different places, while telling us nothing about which one is in which place. Measurements of the two particles' positions are *correlated*. We say that the states of the two particles are *entangled*.

A formula for this entangled wavefunction would have the form

$$\psi(x_1,x_2) = e^{-(x_1-a)^2} e^{-(x_2-b)^2} + e^{-(x_1-b)^2} e^{-(x_2-a)^2}, \tag{3}$$

with the understanding that all positions are measured in suitable units. This formula has the crucial property that it cannot be factored into a function of x_1 times a function of x_2:

$$\psi(x_1,x_2) \neq f(x_1)\cdot g(x_2) \qquad \text{for any functions } f \text{ and } g. \tag{4}$$

By contrast, the Gaussian wavepacket state of equation 2 *can* be factored in this way. The same is true of the other two wavefunctions pictured at the top of the previous page.

Whenever we *can* factor a two-particle wavefunction into a product of single-particle wavefunctions, we can describe the state of either particle without reference to the other. In such cases it is meaningful to say that each particle has its own wavefunction. But the vast majority of all possible two-particle wavefunctions *cannot* be factored; they describe entangled states, for which we cannot accurately describe the state of either particle individually.

So quantum mechanics allows the state of a two-particle system to be entangled. But when do such states actually occur? The answer is: whenever the two particles interact with each other. To be more precise, the potential energy of a two-particle system normally has the form

$$U(x_1,x_2) = U_1(x_1) + U_2(x_2) + U_{12}(x_2 - x_1). \tag{5}$$

The first term represents external forces acting on particle 1; the second term represents external forces acting on particle 2; and the third term represents forces that the two particles exert on each other. *If* the third term is absent, then the Schrödinger equation for the two-particle system splits into separate equations for the two separate particles. In this case we can find a full set of energy eigenstates that are not entangled, and one can prove that if the particles start out in a non-entangled wavefunction, their wavefunction will remain non-entangled (separable) over time. But if the term $U_{12}(x_2 - x_1)$ is present, then the energy eigenstates will generally be entangled, and even if the particles start out in a non-entangled state, their wavefunction will become entangled as time passes.

I've created two web apps that allow you to explore the dynamics that I've just described. "Entanglement in a Box" confines the two particles inside a one-dimensional box and finds the four lowest energy levels and corresponding wavefunctions, with an adjustable attractive or repulsive force between the particles. "Colliding Wavepackets" starts the two particles moving toward each other in non-entangled Gaussian wavepacket states, then shows how they become entangled as they interact via a short-range force. You'll find both simulations at `physics.weber.edu/schroeder/software/`.

Identical particles

Quantum mechanics allows two (or more) particles to be *identical* in a sense that isn't really possible in classical mechanics. Consider what happens when two particles have all the same intrinsic properties (mass, electric charge, spin orientation, etc.), and in addition, their probability density is unchanged under the operation of interchanging its two arguments:

$$|\psi(x_2,x_1)|^2 = |\psi(x_1,x_2)|^2. \tag{6}$$

If this relation holds, then the two particles can't even be distinguished by their locations—or rather, by the probabilities of their being found at various locations—because the probability of finding particle 1 near $x = a$ and particle 2 near $x = b$ is the same as that of finding particle 2 near $x = a$ and particle 1 near $x = b$, for any a and b. This situation differs from classical mechanics, in which you can always distinguish particles by their locations (if nothing else).

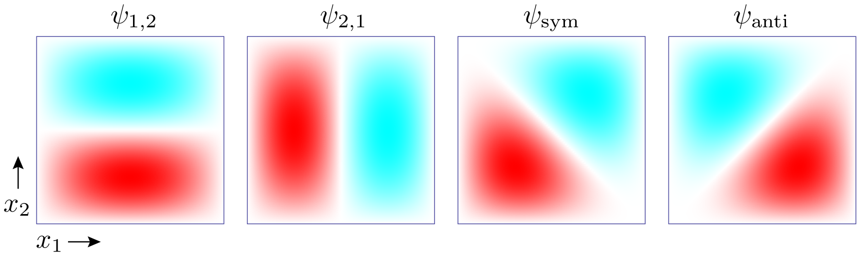
In terms of the wavefunction itself, there are actually two different ways that equation 6 can be realized:

$$\psi(x_2, x_1) = +\psi(x_1, x_2) \quad \text{or} \quad \psi(x_2, x_1) = -\psi(x_1, x_2). \tag{7}$$

In the first case the wavefunction is *symmetric* under interchanging x_1 and x_2, while in the second case it is *antisymmetric.*

It seems that nature has taken advantage of both the symmetric and antisymmetric options, and filled the universe with particles that are truly identical to each other in one or the other of these ways. Particles for which $\psi(x_2, x_1) = +\psi(x_1, x_2)$ are called *bosons* (after Satyendra Nath Bose, 1894–1974), and include photons, pions, helium-4 nuclei, and many other species of nuclei and atoms. Particles for which $\psi(x_2, x_1) = -\psi(x_1, x_2)$ are called *fermions* (after Enrico Fermi, 1901–1954), and include electrons, protons, neutrons, helium-3 nuclei, and many other species of nuclei and atoms. There is a deep connection between the intrinsic spin of a particle and whether it is a boson or a fermion: Particles with integer s values are always bosons, while particles with $s = 1/2$, $3/2$, and so on are always fermions. (This connection is actually a theorem of relativistic quantum field theory, but that's far beyond the scope of this course.)

As a simple illustration, consider these four wavefunctions for a pair of particles in a one-dimensional box:



The first two wavefunctions are simply products of sine waves: $\psi_{1,2} = \sin(\pi x_1)\sin(2\pi x_2)$ and $\psi_{2,1} = \sin(2\pi x_1)\sin(\pi x_2)$. The third and fourth are the symmetric and antisymmetric combinations,

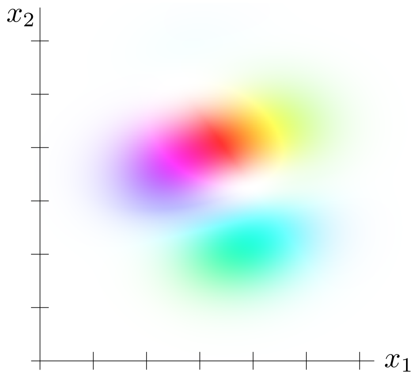
$$\psi_{\text{sym}} = \psi_{1,2} + \psi_{2,1} \qquad \text{and} \qquad \psi_{\text{anti}} = \psi_{1,2} - \psi_{2,1}. \tag{8}$$

If the two particles are distinguishable, then all four of these wavefunctions (and many other mixtures) are allowed. But if the particles are identical bosons, only ψ_{sym} is allowed, because only this mixture is unchanged under $x_1 \leftrightarrow x_2$. And if the particles are identical fermions, then only ψ_{anti} is allowed, because only this mixture changes sign under $x_1 \leftrightarrow x_2$. Notice that $\psi_{\text{anti}} = 0$ along the main diagonal, where $x_2 = x_1$: We'll never find two identical fermions in the same location.

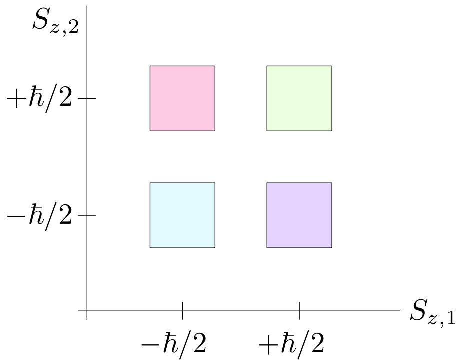
An even simpler example would be if we try to put both particles in the *same* single-particle state: $\psi(x_1, x_2) = \psi_0(x_1)\psi_0(x_2)$, where ψ_0 is any single-particle wavefunction. This two-particle wavefunction is symmetric under $x_1 \leftrightarrow x_2$, so it could describe a system of two identical bosons, or a system of two distinguishable particles, but *not* a system of two identical fermions. It is *impossible* to put two identical fermions into the same single-particle state. This fact is called the *Pauli exclusion principle* (after Wolfgang Pauli, 1900–1958).

Two Spin-1/2 Particles

To describe the quantum state of two structureless particles in one space dimension, we use a *single* wavefunction that's a function of *two*-dimensional configuration space: $\psi(x_1, x_2)$. The value of ψ at any point (x_1, x_2) is a complex number whose square modulus tells us the probability of finding the system in that configuration:



Similarly, to describe the spin portion of the quantum state of two $s = 1/2$ particles, we use a *single* list of complex numbers, which we can think of as a function of *both* particles' potential S_z values. The square modulus of any entry in this list tells us the probability of finding the system with a particular pair of S_z values. This time, though, the list has just four (2×2) values. One way to picture the list is as a two-dimensional density plot over the two discrete S_z variables, with values represented as colors in the usual way:



More often, though, we just list the four complex numbers as a column vector:

$$\chi_{1,2} = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} ++ \\ +- \\ -+ \\ -- \end{pmatrix}, \tag{9}$$

where the second form gives the conventional order for the listing: first the amplitude for finding both spins up ($+\hbar/2$), then the amplitude for finding the first up and the second down ($-\hbar/2$), then the amplitude for finding the first down and the second up, and finally the amplitude for finding both down.

If we know the separate spin states of each of our two particles, then we can construct the system's four-component spin state using what's called a *tensor product*, denoted ⊗:

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a\begin{pmatrix} c \\ d \end{pmatrix} \\ \\ b\begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}. \tag{10}$$

The rule here is to "expand" the spinor on the left, then multiply each of its components by the spinor on the right.

The simplest examples are tensor products of the S_z spin-up and spin-down states:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{11}$$

These are the "up up", "up down", "down up", and "down down" states, respectively.

But the vast majority of all possible states for our two-spin system *cannot* be written as tensor products. For instance, consider the normalized mixture

$$\sqrt{\frac{1}{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \sqrt{\frac{1}{2}}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{1/2} \\ 0 \\ 0 \\ \sqrt{1/2} \end{pmatrix}. \tag{12}$$

Comparing to equation 10, we see that *if* this state were a tensor product, then either a or d would have to be zero. But if a were zero, then the top component of our state would be zero, and it's not; while if d were zero, then the bottom component of our state would be zero, and it's not either. QED, this state can't be factored into a tensor product.

And what's the physical interpretation of the state in equation 12? If we have a two-spin system in this state, and we measure S_z for both particles, we have a 50-50 chance of either finding both with spin up or both with spin down—but zero chance of finding one up and the other down. Isn't that odd? Each of the particles has a 50-50 chance of being (when measured) spin-up or spin-down, but as soon as we determine the spin of one of them, we can be sure that the other will have the same spin. The two spins are entangled.

More generally, any state that can't be factored into a tensor product is entangled. Any state that can be factored into a tensor product is not entangled.

Perhaps the most interesting of all entangled states is the mixture

$$\sqrt{\frac{1}{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \sqrt{\frac{1}{2}}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{1/2} \\ -\sqrt{1/2} \\ 0 \end{pmatrix}, \tag{13}$$

called the *singlet* state. This state is sortof the opposite of the state of equation 12, with a 50-50 chance of getting up-down or down-up if we measure S_z for both particles. That is, we can be sure that the *total* S_z value for the two particles is zero, so we'll always measure one to be up ($+\hbar/2$) and the other to be down ($-\hbar/2$), but we can't predict which will be which, and in fact each particle has a 50-50 chance of coming out up or down.

You may be wondering why I put a relative minus sign between the two terms in equation 13. After all, the interpretation I just gave would work just as well with a plus. Let me explain.

Recall the two-component spinors

$$\chi^{\theta+} = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}, \qquad \chi^{\theta-} = \begin{pmatrix} \sin(\theta/2) \\ -\cos(\theta/2) \end{pmatrix}, \tag{14}$$

which describe states of a spin-1/2 particle whose spin along the θ direction (measured from z toward x) is well defined. Suppose now that we have two particles, one in each of these states. If the first particle is in state $\chi^{\theta+}$ while the second is in state $\chi^{\theta-}$, then the system's four-component spinor state is

$$\chi^{\theta+} \otimes \chi^{\theta-} = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} \otimes \begin{pmatrix} \sin(\theta/2) \\ -\cos(\theta/2) \end{pmatrix} = \begin{pmatrix} \cos(\theta/2)\sin(\theta/2) \\ -\cos^2(\theta/2) \\ \sin^2(\theta/2) \\ -\sin(\theta/2)\cos(\theta/2) \end{pmatrix}. \tag{15}$$

And if it's the other way around, with the first particle in state $\chi^{\theta-}$ and the second in state $\chi^{\theta+}$, then the system's state is

$$\chi^{\theta-} \otimes \chi^{\theta+} = \begin{pmatrix} \sin(\theta/2) \\ -\cos(\theta/2) \end{pmatrix} \otimes \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} = \begin{pmatrix} \sin(\theta/2)\cos(\theta/2) \\ \sin^2(\theta/2) \\ -\cos^2(\theta/2) \\ -\cos(\theta/2)\sin(\theta/2) \end{pmatrix}. \tag{16}$$

Now if we were to add states 15 and 16, we'd get an entangled state with a complicated formula and no particularly special interpretation. But perhaps you can see that if instead we *subtract* them, the top and bottom components will cancel while the two middle components will simplify. In fact the normalized mixture, with a relative minus sign, is

$$\frac{1}{\sqrt{2}}(\chi^{\theta+} \otimes \chi^{\theta-} - \chi^{\theta-} \otimes \chi^{\theta+}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -\cos^2(\theta/2) - \sin^2(\theta/2) \\ \sin^2(\theta/2) + \cos^2(\theta/2) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\sqrt{1/2} \\ \sqrt{1/2} \\ 0 \end{pmatrix}, \tag{17}$$

which (up to a physically irrelevant overall minus sign) is just the singlet state!

Conclusion: We can write the singlet state as an equal mixture of up-down and down-up *along any axis* in the xz plane. (In fact it works along any axis at all, but to prove this we would need to use complex-valued spinors.) Physically, this means that the singlet state has zero total angular momentum along any axis. If a pair of $s = 1/2$ particles is in the singlet state, and we measure the spin of one of them along any axis, then we're guaranteed to find the other one to have the opposite spin if we measure it along the same axis.