Spin-1/2 Quantum Mechanics

These rules apply to a quantum-mechanical system consisting of a single spin-1/2 particle, for which we care only about the “internal” state (the particle’s spin orientation), not the particle’s motion through space.

1. The **state** of the particle is represented by a two-component spinor,

$$\psi = \begin{pmatrix} a \\ b \end{pmatrix},$$

where a and b are complex numbers, normalized so that $|a|^2 + |b|^2 = 1$. Multiplying a spinor by a phase factor $e^{i\phi}$ (such as $e^{i\pi} = -1$ or $e^{i\pi/2} = i$) results in a spinor that represents the same physical state.

2. The observable properties of this system are the values of the angular momentum along any axis: S_x, S_y, S_z, and also any diagonal direction. If we measure one of these quantities, the possible outcomes of the measurement are $+\hbar/2$ and $-\hbar/2$. For each of these values there is a special state-spinor ψ, called an **eigenspinor**, for which the particle has that well-defined value of the measured quantity. The eigenspinor corresponding to the value $+\hbar/2$ is called $\psi_\uparrow$, and the eigenspinor corresponding to the value $-\hbar/2$ is called $\psi_\downarrow$. These two spinors (for any given angular momentum direction) are orthogonal to each other:

$$\psi_\uparrow^* \cdot \psi_\downarrow = 0.$$

(In general, whenever we compute a dot-product of two spinors, we complex-conjugate the one on the left. For instance, the normalization condition in rule 1 can also be expressed by saying that the dot product of any spinor with itself must equal 1.)

3. By convention, the eigenspinors corresponding to well-defined S_z are

$$\psi_{z\uparrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \psi_{z\downarrow} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

while the eigenspinors corresponding to well-defined S_x are

$$\psi_{x\uparrow} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \psi_{x\downarrow} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

4. Suppose now that the particle is in state ψ, and that we perform a measurement of some component of $\vec{S}$, for which the eigenspinors are $\psi_\uparrow$ and $\psi_\downarrow$. Generally, we *cannot* predict whether our measurement will result in the value $+\hbar/2$ or $-\hbar/2$. However, we *can* predict the *probabilities* of getting these two values. Each probability is the square modulus of the component of ψ along the direction of the corresponding eigenspinor. That is, if we write ψ as a linear superposition of the two eigenspinors,

$$\psi = c_\uparrow \psi_\uparrow + c_\downarrow \psi_\downarrow,$$

for some complex coefficients $c_\uparrow$ and $c_\downarrow$, then the probability of getting the result $+\hbar/2$ is $|c_\uparrow|^2$, while the probability of getting the result $-\hbar/2$ is $|c_\downarrow|^2$. Because $\psi_\uparrow$ and $\psi_\downarrow$ are orthogonal to each other, and all spinors are normalized, the sum of these two probabilities will always be 1.

5. Performing a measurement generally alters the state of the particle. Immediately after a measurement is performed, the particle’s spinor will be whatever eigenspinor corresponds to the result that was obtained for the measurement.