Chapter 3

Electron-Positron Scattering

Particle physicists study a bewildering variety of reactions, using a limited but versatile set of experimental and theoretical tools. My approach in this chapter will be to discuss one particular reaction in great detail, using it as a "paradigm" to introduce you to the most important theoretical tools. Rather than trying to rigorously *derive* every formula, I'll merely show how the formulas are used and try to explain why they are plausible. Later, in Chapter 4, I'll present the theoretical tools in more generality, and show how they can be applied to other reactions.

Our paradigm reaction will be **electron-positron scattering**, illustrated in Figure 3.1. It is simply the reaction in which an electron and a positron go in, and an electron and a positron come out (generally in some other direction). While this reaction may not be as glamorous as others in which exotic new particles are created, it has the virtue of being relatively simple, and of involving familiar particles that are easy to detect.

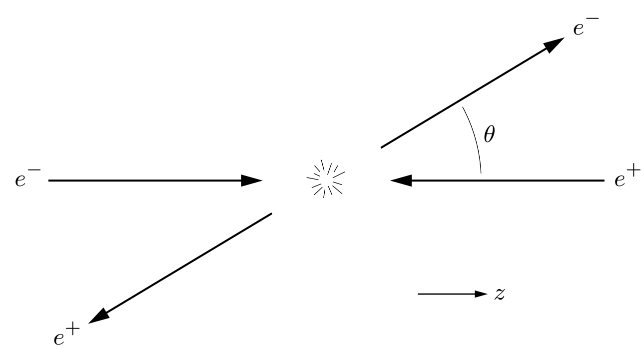


Figure 3.1. An electron-positron (e^-e^+) scattering reaction, viewed in the center-of-mass frame of reference. Note that the z axis is conventionally taken to point along the electron's initial direction of motion, so the scattering angle θ is the same as the usual polar angle in spherical coordinates.

3.1 The Cross Section

For electron-positron scattering, like most other reactions, the comparison of theory to experiment centers on the cross section. Experimental physicists measure the cross section (dividing the observed event rate by the known luminosity of the collider), while theoretical physicists predict the cross section. More precisely, the quantity of interest is the differential cross section, $d\sigma/d\Omega$: the function that, when integrated over any range of θ and ϕ, yields the cross section for scattering into that range.

For simplicity, let's assume that our two beam energies are equal (as they are at most, but not all, colliders). If E denotes the energy of either beam, the total energy going into the reaction is

$$E_{\rm cm} = 2E, \tag{3.1}$$

where the "cm" subscript indicates that this is the total energy as measured in the center-of-mass reference frame (which is the same as the laboratory frame). Because $E_{\rm cm}$ is generally in the GeV range, it is almost always valid to work in the limit where the electron mass m (about half an MeV) is negligible compared to E or $E_{\rm cm}$. In this limit, we can treat as a massless particle traveling at the speed of light, and the differential cross section will be independent of m.

Another helpful simplification is to assume that the electron and positron beams are unpolarized, that is, that their spin orientations are random. This is true at some colliders but not all. In addition, we generally assume that the polarizations of the outgoing particles are not measured, so that σ is the cross section for the sum of all possible outgoing polarizations.

The unpolarized cross section for electron-positron scattering in the center-of-mass frame, in the limit $E \gg m$, can depend on only two variables: the beam energy E and the scattering angle θ. Furthermore, the dependence on E is uniquely determined by dimensional analysis. The cross section must have units of area, or length squared, and the only length that we can make from E and θ is the particles' de Broglie wavelength,

$$\lambda = \frac{h}{p} = \frac{hc}{E}. \tag{3.2}$$

Therefore the cross section must be proportional to λ^2, or E^{-2}:

$$\frac{d\sigma}{d\Omega} = \frac{1}{E^2} \times (\text{some function of } \theta). \tag{3.3}$$

Experiments have confirmed this prediction over a wide range of energies. Loosely speaking, the higher the energies of the incoming particles, the more likely they are to miss each other.

The θ dependence of the differential cross section is much more complicated, and therefore more interesting. Figure 3.2 shows a set of experimental measurements of the event rate for this reaction, taken at the SPEAR collider at Stanford during the 1970's, at $E_{\rm cm} = 4.8$ GeV. As you can see, the number of events (and hence the differential cross section) increases as θ decreases, and goes to infinity in the limit $\theta \to 0$.

The solid curve in Figure 3.2 shows a theoretical prediction for the electron-positron scattering cross section, first derived by H. J. Bhabha in 1935. Bhabha's theoretical formula is

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{32\pi^2 E_{\rm cm}^2}\left[\frac{1+\cos^4\frac{\theta}{2}}{\sin^4\frac{\theta}{2}} - \frac{2\cos^4\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} + \frac{1+\cos^2\theta}{2}\right], \tag{3.4}$$

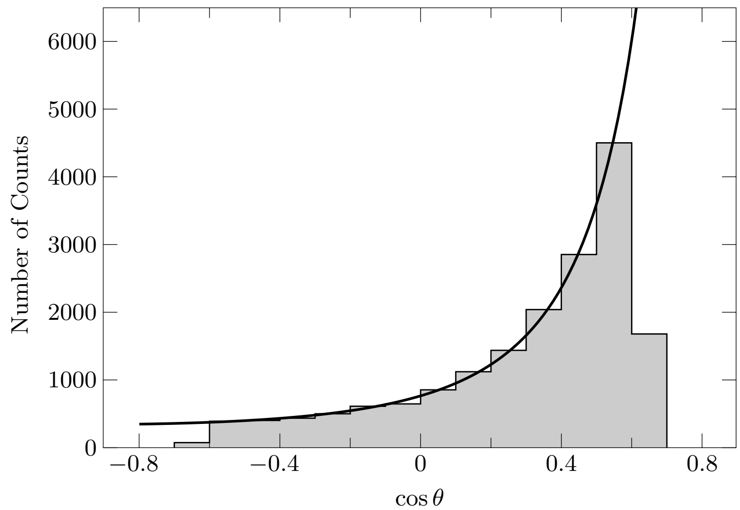


Figure 3.2. Measurements of the electron-positron scattering event rate at $E_{\rm cm} =$ 4.8 GeV, taken at Stanford's SPEAR collider in the early 1970s. The horizontal axis is $\cos\theta$, so θ increases from right to left. The data are grouped into bins, each covering a range of $\cos\theta$ values with a width of 0.1. The sharp cutoff in the data at $|\cos\theta| > 0.6$ is caused by the shape of the detector. The smooth curve shows the angular dependence predicted by the Bhabha formula, with the vertical scale adjusted to give a good fit to the data. Adapted from J.-E. Augustin et al., Measurement of $e^+e^- \to e^+e^-$ and $e^+e^- \to \mu^+\mu^-$, *Phys. Rev. Lett.* **34**, 233–236 (1975).

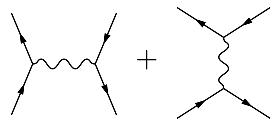
where e is the strength of the electron's charge. Even though the curve looks simple, the formula is not simple at all! Fortunately, though, the formula has a simple interpretation, to which we now turn.

Problem 3.1. In Figure 3.2, the experimental data are "binned" into equal-width intervals of $\cos\theta$ (rather than θ itself). Show that, in the limit where the bin widths are infinitesimal, this binning should produce a function that is proportional to $d\sigma/d\Omega$. (Assume that the differential cross section is independent of the azimuthal angle, ϕ.)

Problem 3.2. Starting from Bhabha's formula, find the total cross section for electron-positron scattering at $\theta > 90^\circ$, in terms of $E_{\rm cm}$ and fundamental constants. Write your answer in terms of the fine-structure constant $\alpha = e^2/4\pi\epsilon_0\hbar c$, and restore whatever other factors of $\hbar$ and c are necessary to obtain an answer in SI units (m^2). Evaluate your result numerically at $E_{\rm cm} = 5$ GeV and at $E_{\rm cm} = 100$ GeV.

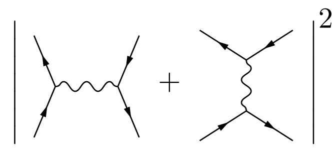
3.2 Feynman Diagrams

The interpretation of Bhabha's formula is a pictorial one, invented by Richard Feynman in the late 1940's. Aside from a few multiplicative factors that I'll discuss later, Bhabha's formula is equal to the square modulus of a quantum mechanical amplitude (a complex number), and this amplitude is equal to the sum of two terms, each represented by a Feynman diagram:

Amplitude =  . (3.5)

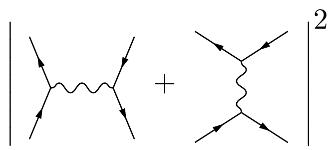
The first term in Bhabha's formula comes from the square of the first diagram, the last term comes from the square of the second diagram, and the middle term comes from the cross term (or "interference term") between the two diagrams.

So the diagrams are really a pictorial way of remembering the formula for the quantum mechanical amplitude for this reaction. But they also provide us with an interpretation of the formula. Each straight line in the diagrams represents an electron (if the arrow points up) or positron (if the arrow points down), with the initial-state particles at the bottom of the diagram and the final-state particles at the top. The wavy lines represent photons, which in this case are called "virtual" particles, since they appear as intermediate quantum mechanical states, but are never actually observed. In the first diagram, the virtual photon is exchanged between the electron and the positron, while in the second diagram, the electron and positron annihilate to form a virtual photon, which converts into another electron and positron. Thus, electron-positron scattering can occur either via exchange or via annihilation, and both of these processes contribute to the overall probability. But remember the quantum mechanical rule: To compute the probability (or cross section), we *first* add the separate amplitudes together, *then* square the result:

$\frac{d\sigma}{d\Omega} \propto$  . (3.6)

The total probability is *not* simply the sum of the exchange probability plus the annihilation probability—there is also the cross term, due to quantum interference of the exchange and annihilation amplitudes.

If we include the factors omitted from the proportionality in equation 3.6, the full formula turns out to be

$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{\mathrm{cm}}^2}$  . (3.7)

Comparing to equation 3.3, we therefore see that the diagrams themselves must evaluate to unitless numbers. More importantly, though, each diagram must evaluate to a Lorentz-invariant quantity, while the cross section, being an effective area in the xy plane, is not Lorentz-invariant. The $1/E^2$ in the prefactor gives the cross section the correct units and the correct properties under Lorentz transformations; I'll say more about where it comes from in the following chapter. The factor of $64\pi^2$ is purely conventional, but has the virtue of simplifying the Feynman rules below.

Problem 3.3. Plot the θ-dependent part of Bhabha's formula vs. $\cos\theta$, over the same range of angles shown in Figure 3.2. (Hint: First use trig identities to rewrite $\sin^2\frac{\theta}{2}$ and $\cos^2\frac{\theta}{2}$ in terms of $\cos\theta$.) On the same graph, plot the sum of the first and last terms of the formula, omitting the interference term. Discuss whether the data in Figure 3.2 is sufficiently accurate to detect the quantum interference between the exchange and annihilation amplitudes.

3.3 Feynman Rules

To associate each diagram with an actual formula for the corresponding amplitude, we use what are called the **Feynman rules**. The rules tell us to write a brief formula for each part of a diagram (each external line, each internal line, and each vertex), then multiply these expressions together to obtain the value of the whole diagram. This procedure is an example of the general quantum mechanical method for computing the amplitude of a multi-step process: Multiply the amplitudes for the subprocesses to obtain the amplitude for the whole process.

But before writing down the Feynman rules, I need to make another simplification: I'll neglect the fact that both electrons and photons carry spin angular momentum, which can point in various directions. Pretending that all particles have spin zero will simplify the Feynman rules considerably, allowing us to concentrate on the essential features dictated by quantum mechanics and special relativity. Of course, this simplification will also affect the final answer for the cross section; I'll describe these effects later.

Let's start with the Feynman rules for the exchange diagram. Figure 3.3 shows the diagram, with the amplitude for each of its pieces written alongside. In the diagram I've labeled each vertex with a four-vector position in spacetime, x or y. I've also labeled each external line with a four-vector momentum. (Later we'll express these momenta in terms of their components in our coordinate system, for instance, $p_1 = (E, 0, 0, E)$ for the massless ingoing electron.) Notice that the arrows on the diagram itself do *not* necessarily indicate the directions of the momentum vectors, so I've added arrows alongside to remind us that all momenta point upward, forward in time.

Although we *could* just take the Feynman rules as given, and proceed to calculate the amplitude for the exchange diagram, I'd like to digress for a moment to discuss the meanings of the various Feynman rule expressions, and where they come from. Notice that the rules include three basic types of expressions, corresponding to the diagram's external particles, virtual particles, and vertices.

External particles. The Feynman rule for an *initial*-state particle should be the amplitude for a particle with a certain momentum to be found at a certain point. For example, for the initial electron, we want the amplitude for a particle with four-momentum p_1 to be found at the spacetime point x. But this amplitude is simply the electron's (time-dependent) wavefunction, $e^{-i\omega t}e^{i\vec{p}_1\cdot\vec{x}} = e^{-ip_1\cdot x}$. Similarly, the amplitude for the initial positron to be found at point y is $e^{-ip_2\cdot y}$. For a *final*-state particle, we want the reverse: the amplitude for a particle at a given spacetime point to be found with the desired four-momentum vector. According the the usual

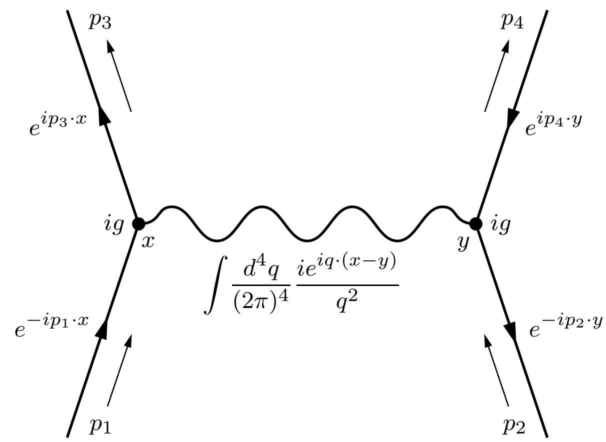


Figure 3.3. Feynman rules for translating the exchange diagram into a quantum mechanical amplitude, treating all particles as if they were spinless. The labels x and y represent the four-vector positions of the vertices, while the various p's label the four-vector momenta of the ingoing and outgoing particles. The remaining expressions ($e^{-ip_1\cdot x}$, ig, etc.) represent the Feynman amplitudes for the components of the diagram.

quantum-mechanical rule, this reversal is given by the complex conjugate. For instance, for the final electron, the amplitude is $(e^{-ip_3\cdot x})^* = e^{+ip_3\cdot x}$.

Virtual particles. The Feynman rule for a virtual particle represents the amplitude for the particle to propagate from one spacetime point to another. In general, for a spinless particle, this amplitude is

$$D_F(x-y) = \int \frac{d^4q}{(2\pi)^4}\frac{ie^{iq\cdot(x-y)}}{q^2-m^2}, \tag{3.8}$$

an expression known as the **Feynman propagator**. Here m is the mass of the virtual particle, which for a photon is zero.

How can we understand this expression? One way is to interpret q as the four-momentum vector of the virtual photon. The factor $e^{iq\cdot x}$ is the amplitude for a photon at point x to be found with momentum q, while the factor $e^{-iq\cdot y}$ is the amplitude for a photon with momentum q to be found at point y. We integrate over all q vectors because quantum mechanics tells us to sum over all the alternative ways in which the event can occur. The factor $1/q^2$ (or $1/(q^2-m^2)$ for a massive particle) must then be the amplitude for a virtual particle to "exist" with four-momentum q. Notice that this factor would be *infinite* for a real particle, for which q^2 must equal m^2 by the relativistic energy-momentum relation. The closer the virtual particle is to being real, the larger its amplitude to exist.

We can also think of the Feynman propagator as the quantum mechanical *wavefunction* of a particle produced at a point, propagating outward according to the Klein-Gordon equation. For definiteness, let's imagine that x is the fixed point where the particle is created, while y is the variable point that is the argument of the wavefunction. (Interchanging x with y leaves the propagator unchanged, so we could just as well imagine the opposite.) You can easily check (see Problem 3.4) that $D_F(x-y)$ satisfies the Klein-Gordon equation everywhere except at $y=x$. *At* $y=x$, we would have to add an infinite (delta function) potential term to the Klein-Gordon equation, to represent the disturbance that creates the particle. This infinite potential term results in an infinite wavefunction at $y=x$, and indeed, $D_F(x-y)$ is infinite there.

Vertices. The Feynman rule for each vertex in a diagram is simply ig, where g is called a **coupling constant**, a number that we must measure. For the actual electron-photon vertex, the coupling constant would be the magnitude of the electron's charge, e, a unitless number (in units where $\hbar = c = \epsilon_0 = 1$). For spinless particles, however, the coupling constant must have units of mass (or momentum or energy). In any case, the coupling constant determines the "strength" of the interaction that allows an electron to emit or absorb a photon.

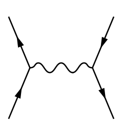
If we multiply together all the expressions shown in Figure 3.3, we obtain the amplitude for the exchange process to occur at a *particular* pair of spacetime points x and y. But these two points could be *anywhere*, so to obtain the *total* amplitude for the exchange process, we should integrate over all x and y. These integrations are just another example of summing over all of the alternative ways in which an event can happen.

Problem 3.4. Show that the Feynman propagator, considered as a function of y with x fixed, is a solution to the Klein-Gordon equation everywhere except at $y=x$. Show that the Feynman propagator is infinite at $y=x$.

Problem 3.5. Show that the Feynman propagator is unchanged if you interchange x and y. Discuss whether it is meaningful to ask which direction the virtual photon travels.

3.4 The Calculation

Applying all the Feynman rules just discussed gives us the total amplitude for the exchange process (still neglecting spin):



$$= \int d^4x \int d^4y \; (ig)^2 \, e^{-ip_1\cdot x} e^{-ip_2\cdot y} e^{ip_3\cdot x} e^{ip_4\cdot y} \int \frac{d^4q}{(2\pi)^4} \frac{ie^{iq\cdot(x-y)}}{q^2}. \tag{3.9}$$

Fortunately, it isn't hard to simplify this expression.

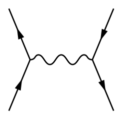
First notice that x appears only in the exponential factors, so the integral over x yields a four-dimensional delta function:

$$\int d^4x \, e^{-ip_1\cdot x} e^{ip_3\cdot x} e^{iq\cdot x} = \int d^4x \, e^{i(p_3+q-p_1)\cdot x} = (2\pi)^4 \, \delta^4(p_3+q-p_1). \tag{3.10}$$

If we interpret q as the momentum of the virtual photon, then this delta function tells us that the amplitude is nonzero only when the emission of this photon (at point x) conserves momentum. Similarly, the integral over y is

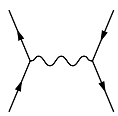
$$\int d^4y \, e^{-ip_2\cdot y} e^{ip_4\cdot y} e^{-iq\cdot y} = \int d^4y \, e^{i(p_4-p_2-q)\cdot y} = (2\pi)^4 \, \delta^4(p_4-p_2-q), \tag{3.11}$$

telling us that momentum must be conserved when the virtual photon is absorbed at point y. With both the x and y integrals carried out, the exchange amplitude becomes



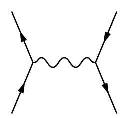
$$= -ig^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2} (2\pi)^4 \delta^4(p_3 + q - p_1) (2\pi)^4 \delta^4(p_4 - p_2 - q). \qquad (3.12)$$

We can now use either delta function to carry out the q integral. Let's use the first delta function, which instructs us to set q equal to $p_1 - p_3$ everywhere else in the integrand:



$$= -ig^2 \frac{1}{(p_1 - p_3)^2} (2\pi)^4 \delta^4(p_3 + p_4 - p_1 - p_2). \qquad (3.13)$$

The remaining delta function now expresses *overall* four-momentum conservation, and by convention, this delta function is actually *not* included in the final amplitude. We therefore have simply

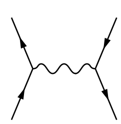


$$= \frac{-ig^2}{(p_1 - p_3)^2}. \qquad (3.14)$$

Because equation 3.14 is written in terms of four-vectors, we can see that the exchange amplitude is invariant under Lorentz transformations. But *we* want an answer that is written in terms of the beam energy E and scattering angle θ, as measured in the center-of-mass reference frame. Figure 3.4 shows how p_1 and p_3 are related to E and θ. The denominator of the exchange amplitude is therefore

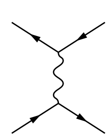
$$\begin{aligned}
(p_1 - p_3)^2 &= p_1^2 + p_3^2 - 2p_1 \cdot p_3 \\
&= 0 + 0 - 2(E, 0, 0, E) \cdot (E, E\sin\theta, 0, E\cos\theta) \\
&= -2E^2(1 - \cos\theta) \\
&= -E_{\text{cm}}^2 \sin^2 \tfrac{\theta}{2},
\end{aligned} \qquad (3.15)$$

where in the last line I've used a standard trig identity and $E_{\text{cm}} = 2E$. In terms of E_{cm} and θ, then, the exchange diagram is



$$= \frac{ig^2}{E_{\text{cm}}^2 \sin^2 \frac{\theta}{2}}. \qquad (3.16)$$

A very similar calculation (see Problem 3.8) gives for the annihilation amplitude



$$= -\frac{ig^2}{E_{\text{cm}}^2}, \qquad (3.17)$$

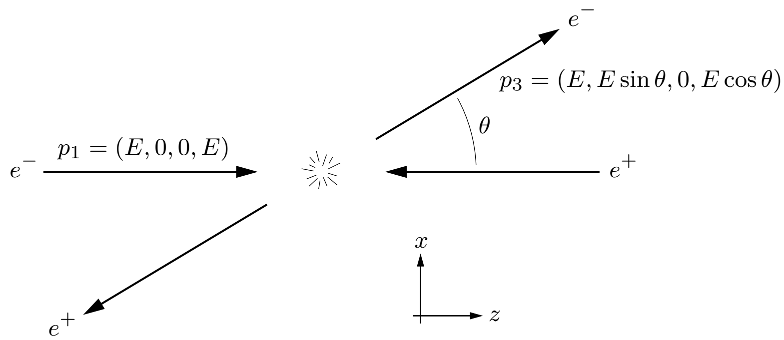
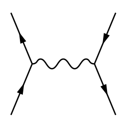


Figure 3.4. For electron-positron scattering in the limit $E \gg m$, the electron's initial and final four-momentum vectors can be written in terms of E and θ as shown.

independent of θ. Adding this to the exchange amplitude, taking the square modulus, and dividing by $64\pi^2 E_{\rm cm}^2$ (according to equation 3.7), we finally obtain for the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{\rm cm}^2}\left| \quad\right.$$



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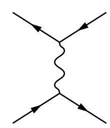

$$\left. \quad \right|^2 \tag{3.18}$$

$$= \frac{1}{64\pi^2 E_{\rm cm}^2}\left(\frac{g^2}{E_{\rm cm}^2}\right)^2 \left[\frac{1}{\sin^4\frac{\theta}{2}} - \frac{2}{\sin^2\frac{\theta}{2}} + 1\right].$$

If electrons, positrons, and photons were truly spinless, then we would expect this formula to agree with experimental measurements of the electron-positron scattering rate.

Comparing our final result with Bhabha's prediction for *real* electron-positron scattering (equation 3.4), you can see that there are some similarities and some differences. Bhabha's formula is proportional to e^4, where e is the strength of the electron's charge, but our result instead contains g^4, where g is some fictitious coupling constant. It's tempting to simply set $g = e$, but we can't do that because g has units of mass (or momentum or energy); you can see this from the four extra powers of $E_{\rm cm}$ in the denominator of our formula. This incorrect energy dependence is a major embarassment, and I'll discuss it further in the following section. As for the angular dependence of our result, it is obviously not the same as in Bhabha's formula, but notice that the differences are entirely in the *numerators* of the three terms (which I'll also discuss below). The *denominators*, which come from the $1/q^2$ in the photon propagation amplitudes, are correct! Note in particular that the first term is proportional to $(\sin\frac{\theta}{2})^{-4}$, which diverges as $\theta \to 0$. This dependence is the dominant feature of the cross section at small angles, and agrees with experiment. It is a general feature of scattering due to a Coulomb $1/r^2$ force, for instance, in the classical Rutherford formula for scattering of an alpha particle from a heavy nucleus.

Problem 3.6. To carry out the q integral in the equation 3.12, I used the first delta function. Show that if you instead use the second delta function, you obtain the same final result.

Problem 3.7. In the first step of equation 3.15, I squared the four-vector difference $p_1 - p_3$ before expressing p_1 and p_3 in terms of components. Show that you get the same result if you first subtract the four-vectors component by component, then take the square modulus of the resulting four-vector.

Problem 3.8. In this problem you will evaluate the annihilation diagram, in analogy to the evaluation of the exchange diagram presented above.

(a) Label the annihilation diagram with x's and p's, and for each part of the diagram, write the appropriate Feynman amplitude, in analogy to Figure 3.3. Notice that the plane-wave amplitudes for the external legs of the diagram associate position vectors with momentum vectors in a way that is different from the exchange diagram.

(b) Write down the total amplitude for the annihilation process, and simplify it to obtain equation 3.17. During the calculation, pause to interpret each delta function verbally.

(c) Would it be correct to say that during the annihilation process, *first* the initial electron and positron annihilate into a virtual photon, *then* the virtual photon converts into an electron-positron pair? Discuss.

(d) Fill in the remaining steps to derive the predicted cross section, equation 3.18.