Electromagnetic Theory
Fall 2019

Problem Set 7

(due Monday, October 28, 4:00 pm)

1. Recall the argument given in class for why length contraction results in velocity-dependent forces between moving charges. Consider a similar scenario that is the same in every way except that the test charge is moving in the opposite direction. So in the lab frame, the positive charges in the wire are at rest, the negative charges in the wire are moving to the right at speed v, and the test charge is moving to the *left* at speed v. In this frame, the average distance between the positive charges in the wire is ℓ, and the average distance between the negative charges is also ℓ.

   (a) What are the velocities of the various charges in the "test charge" frame, in which the test charge is at rest? (To find the velocity of the negative charges in the wire, you'll need to use the relativistic velocity transformation formula. The answer is *not* $2v$!)

   (b) Find the average distance ℓ_+ between the positive charges in the test charge frame, in terms of ℓ and v.

   (c) Find the average distance ℓ_- between the negative charges in the test charge frame, in terms of ℓ and v, and simplify your formula as much as possible. (Hint: In order to use the length contraction formula you will need to relate ℓ_- to the average distance between these charges in a *third* reference frame, in which they are at rest. Be sure to check your work in whatever ways you can think of. It is not necessary to use the binomial expansion.)

   (d) Find a formula for the linear charge density of the wire, and simplify your result in the limit $v \ll c$.

   (e) Finally, determine the magnitude and direction of the force on the test charge, and explicitly identify the formula for the magnetic field created by the wire in the lab frame.

2. Find the density (charge per unit volume) of conduction electrons in a piece of copper, assuming that each atom contributes one conduction electron. (Hint: You can look up various properties of copper. What properties do you need to know? Be sure to state clearly what values you looked up, and where.) Now suppose you have a copper wire that is 1 mm in diameter, carrying a current of 1 A. What is the average velocity of the conduction electrons? Discuss your result briefly: Why do the lights come on instantly when you flip the switch? How can relativistic length contraction possibly account for magnetic forces?

3. In 1897, J. J. Thomson "discovered" the electron by measuring the charge-to-mass ratio of "cathode rays" (streams of electrons in a vacuum tube).

   (a) First he passed the beam through electric and magnetic fields that were perpendicular to each other and to the beam, and adjusted the electric field until he got zero deflection. For this situation, find the speed of the particles in terms of $|\mathbf{E}|$ and $|\mathbf{B}|$.

(b) Then he turned off the electric field, and measured the beam's radius of curvature, R, when it was deflected by the magnetic field alone. For this situation, derive the relation between R and $|\mathbf{B}|$, and use this relation to find an expression for the charg-to-mass ratio, q/m, in terms of $|\mathbf{E}|$, $|\mathbf{B}|$, and R.

(c) Today, of course, you can look up the values of q and m. Use these known values, and your laboratory experience and other knowledge, to find a reasonable and consistent set of values for R, $|\mathbf{E}|$, $|\mathbf{B}|$, and v. Briefly explain why your values are reasonable.

4. (a) A phonograph record carries a uniform surface charge density σ. If it rotates at angular velocity ω, what is the surface current density at a distance r from the center? (b) A uniformly charged solid sphere, of radius R and total charge Q, is centered at the origin and spinning at a constant angular velocity ω about the z axis. Find the volume current density $\mathbf{J}$ at an arbitrary point (r, θ, ϕ) within the sphere.

5. Use Mathematica to produce a vector plot of the magnetic field created by a long, straight current-carrying wire. Take the wire to stretch along the z axis with the current in the $+z$ direction, and plot the field in the xy plane. Use units in which $\mu_0 I/(2\pi) = 1$. Before you start typing code into Mathematica, be sure that you have the correct formulas for B_x and B_y in rectangular coordinates.

6. Use Mathematica to produce a vector plot of the magnetic field of a *pair* of long, straight current-carrying wires, both parallel to the z axis. One wire passes through the point $(1, 0, 0)$ and carries a current running in the $+z$ direction; the other passes through the point $(-1, 0, 0)$ and carries an equal-magnitude current running in the $-z$ direction. Plot the field in the xy plane.

7. (a) Find the magnetic field at the center of a square loop that carries a steady current I, in terms of the distance from the center to the middle of one side (call this distance R). (b) Now replace the square with a regular n-sided polygon, still carrying a current I, with the same distance R from the center to the middle of any side. Again find a formula for the magnetic field at the center. (c) Take the limit of your previous result in which the number of sides goes to infinity, and check that you get the expected result.

The rotating armatures of every generator and every motor in this age of electricity are steadily proclaiming the truth of the relativity theory to all who have ears to hear.

—Leigh Page