Electromagnetic Theory
Fall 2019

# Problem Set 5

(due Monday, October 7, 4:00 pm)

1. Imagine a very long conducting cylinder of radius $a$, centered inside a long, hollow conducting cylinder of radius $b$. (A real-world example would be a coaxial cable.) Find the capacitance per unit length of this pair of conductors, in terms of $a$ and $b$. Briefly discuss whether your result is reasonable: About how much capacitance does a typical coaxial cable have? What happens in the limit where $b$ and $a$ are almost equal?

2. See the accompanying page for a short Mathematica program that solves Laplace's equation in two dimensions on a $50 \times 50$ grid using the relaxation method. Carefully type this code into Mathematica, thinking about each instruction as you type it. (If you're a slow typist you may skip the comments.) Notice that the code is organized into four parts. Parts 1 and 2 initialize the system, while Part 3 handles the graphics. Execute these three parts in succession, and make sure you understand what they're doing (though you may gloss over the arcane list manipulation in the first line in Part 3 and the similar functions in the following line). Then execute both cells in Part 4, which implements the relaxation algorithm. You should see the graphics display update as the algorithm proceeds. Execute the final cell again as needed until the image stops changing. Once you have everything working, answer the following questions.

   (a) Briefly explain the line of code that computes the nearest-neighbor average to implement the actual relaxation calculation. Write out this line in more readable math notation.

   (b) What is the purpose of the `If` function that surrounds the relaxation calculation formula? Explain carefully.

   (c) Why do the `For` loops in the `relaxStep` definition start at 2 rather than 1, and keep `x` and `y` strictly less than their maximum values, rather than less than or equal to?

   (d) Carefully explain the formula that calculates the electric field from the potential.

   (e) Comment-out the line that creates `vPlot` as a `ListDensityPlot`, and replace it with a similar line that makes `vPlot` a `ListContourPlot` instead. You won't need the `InterpolationOrder` option, but you should set the `Contours` option to something higher than the default.

   (f) Insert a `VectorScale` option into the line that creates `ePlot`, in order to omit the largest electric field vectors and, hence, enlarge the rest of the vectors so they'll be more visible.

   (g) When you are happy with the appearance of the plot, execute a few hundred more relaxation steps to be sure that the calculation has converged. Then print your code and plot, to turn in.

3. Use the relaxation code of the previous problem to find the potential and field around a pair of linear electrodes, each running from $x = 15$ to $x = 35$. One electrode should be at $y = 20$ and $V = -10$, while the other should be at $y = 30$ with $V = +10$. I suggest using a `For` function to loop over these grid locations. Execute enough relaxation steps for the plot to no longer changes visibly. About how many steps were required? Turn in a printout of your code and of the final contour/vector plot. Describe the notable features of the resulting potential and field, especially in the space between the two electrodes.

4. Repeat the previous problem, but modify the positive electrode to make it V-shaped. The bottom vertex should be at $x = 25$ and $y = 30$, and from there the electrode should slant upward at 45-degree angles, about 10 units both horizontally and vertically, to each side. Again execute relaxation steps until the plot remains unchanged, turn in a printout of your modified code and plot, and describe the notable features of this potential and field configuration. What's happening near the vertex of the V, and what does this say about the distribution of charge on the electrode?

5. Use Mathematica to study the final solution to the "infinite slot" scenario, Example 3.3, with a constant $V_0$. First use Mathematica to prove that the solution, Equation 3.37, satisfies Laplace's equation. Then make a contour plot of the potential, combined with a vector plot of the electric field. (Use units in which $V_0 = 1$ and $a = 1$.) As always, spend some time adjusting the plot options to bring out the important features. Discuss these features briefly. Finally, modify your relaxation code to model this scenario, and turn in a plot of the result (obviously you can't allow $x$ to go to infinity, but it suffices to make `xMax` two or three times larger than `yMax`.

6. Suppose that you want to use the relaxation method to study not just conductors held at fixed $V$ values, but also to include fixed charges at specified locations. Then you get to solve Poisson's equation, with a nonzero $\rho$, instead of Laplace's equation. What, then, should be the relaxation formula that forms the core of the algorithm? Modify your Mathematica relaxation code to implement your new formula, by including a `rhoArray` list in the initializations. Then use your code to calculate and plot the potential and field for a single positive point charge located at $y = 10$, reasonably close to the bottom edge of the region which is held at $V = 0$. Make sure all the other edges of the region are a few times farther away from the point charge, so their effects will be small. Comment on the features of the solution.