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A general principle for light reflecting from a uniformly moving mirror: A relativistic treatment

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We investigate the reflection of light from a mirror moving at relativistic speeds and introduce a general principle to determine the relationship between the incident and reflected angles. This principle states that the momentum change of each photon is perpendicular to the surface of the mirror. We carry out sample calculations for various geometries. © 2012 American Association of Physics Teachers.
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I. INTRODUCTION

The study of light reflecting from a surface such as mirror is usually carried out by applying the appropriate boundary conditions for the incident and reflected electromagnetic waves at the reflecting surface.[1] The result is the well-known law of reflection: the angle of incidence is equal to the angle of reflection. Here, the underlying assumption is that all measurements are performed in the rest frame of the source, the reflecting surface, and the detector. However, if the same phenomenon is studied from the point of view of the lab frame, where the source, the reflecting surface, and the detector are all set in motion with a uniform velocity, the well-known law of reflection is no longer valid. In fact, it is well established that for light reflected from a mirror moving at relativistic speeds, the angle of reflection is not necessarily equal to the angle of incidence.[2–10] In this paper, we propose that a basic principle of specular reflection is that the momentum change for reflecting light must be perpendicular to the plane of the mirror. This principle applies to stationary mirrors as well as moving mirrors. Using this principle, the relationships between incident and reflected angles can readily be determined for any mirror speed and any particular geometry.

II. MOMENTUM CHANGE UPON REFLECTION

A. A stationary mirror

Consider light reflecting from a stationary mirror as shown in Fig. 1. We shall analyze this from the quantum viewpoint, in which we treat light as photons. We assume that for each reflected photon, the momentum change $\Delta\vec{P}$ is perpendicular to the plane of the mirror and thus lies along the normal line. We further assume that, for each reflected photon, the incident momentum $\vec{P}_b$ (momentum before) and the reflected momentum $\vec{P}_a$ (momentum after) are equal in magnitude; energy is conserved and each reflected photon undergoes elastic scattering. Because $P_a = P_b$ and $\Delta\vec{P} = \vec{P}_a - \vec{P}_b$, the geometry in Fig. 1 shows a parallelogram of four equal sides, bisected by $\Delta\vec{P}$. The two enclosed triangles are therefore identical, showing that the angle of incidence α and the angle of reflection β are equal in the rest frame. This is normally an assumed "law" for reflection, but here we have shown that it follows from the more general principle that the momentum change vector is perpendicular to the surface of the mirror.

Further, if the momentum change is perpendicular to the mirror, there is no change in momentum tangent to the mirror surface. Thus, the tangential component of $\vec{P}_a$ is equal to the tangential component of $\vec{P}_b$ and there is no reflection-induced stress in the plane of the mirror (consistent with the photon undergoing elastic scattering).

B. A moving mirror

If we now consider the same experiment as viewed by an inertial observer in relative motion to the mirror, $\Delta\vec{P}'$ (measured in the frame in which the mirror is moving) must again be perpendicular to the reflecting surface. Why? Because if $\Delta\vec{P}'$ has a non-zero tangential component that depends on the speed of the mirror, it would be possible for observers watching the moving mirror to measure an associated tangential stress that is not present in the rest frame. Such a scenario would violate the basic principle of relativity and provides a qualitative understanding for why the momentum change for reflected photons must lie along the normal to the mirror. We will prove this statement mathematically in Sec. III.

III. DERIVATION OF MOMENTUM CHANGE DIRECTION

Consider a mirror of length m that is at rest in the xy-frame (Fig. 1), but moving horizontally with a constant speed v as seen in the $x'y'$-frame (Fig. 2). Note that the rest frame angles α and ϕ are arbitrary so that the derivation holds for light incident at any angle on a mirror at any orientation with respect to the horizontal axis. The goal is to show that in the lab frame, the change of momentum $\Delta\vec{P}'$ is perpendicular to the plane of the mirror designated by the mirror vector $\vec{M}'$. That is, we want to prove that $\Delta\vec{P}'\cdot\vec{M}' = 0$.

We begin by transforming the momentum components of the incident and reflected photons from the xy-frame to the $x'y'$-frame. Applying the Lorentz transformations, we get

$$P'_{bx} = \gamma\left(P_{bx} + \frac{v}{c}P_b\right) = \gamma P_b\left[\sin(\alpha-\phi) + \frac{v}{c}\right], \quad (1)$$

$$P'_{by} = P_{by} = P_b\cos(\alpha-\phi), \quad (2)$$

$$P'_{ax} = \gamma\left(P_{ax} + \frac{v}{c}P_a\right) = \gamma P_a\left[\sin(\beta+\phi) + \frac{v}{c}\right], \quad (3)$$

$$P'_{ay} = P_{ay} = -P_a\cos(\beta+\phi), \quad (4)$$

where $\gamma = 1/\sqrt{1-v^2/c^2}$ and c is the speed of light. Note that in the mirror's rest frame we have $P_a = P_b = P$, which allows us to write the momentum vectors in the $x'y'$ frame as

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$$\vec{P}'_b = \gamma P\left[\sin(\alpha-\phi)+\frac{v}{c}\right]\hat{i} + P\cos(\alpha-\phi)\hat{j}, \tag{5}$$

$$\vec{P}'_a = \gamma P\left[\sin(\beta+\phi)+\frac{v}{c}\right]\hat{i} - P\cos(\beta+\phi)\hat{j}. \tag{6}$$

But we already know that $\beta = \alpha$ in the rest frame, which allows us to calculate the momentum change in the lab frame as

$$\Delta\vec{P}' = P\gamma[\sin(\alpha+\phi)-\sin(\alpha-\phi)]\hat{i} - P[\cos(\alpha+\phi)+\cos(\alpha-\phi)]\hat{j}, \tag{7}$$

which simplifies to

$$\Delta\vec{P}' = 2P\cos\alpha\left(\gamma\sin\phi\hat{i} - \cos\phi\hat{j}\right). \tag{8}$$

Now let us transform the mirror vector $\vec{M}'$ to get

$$\vec{M}' = \frac{m\cos\phi}{\gamma}\hat{i} + m\sin\phi\hat{j}. \tag{9}$$

Forming the dot product of Eqs. (8) and (9) then gives

$$\Delta\vec{P}'\cdot\vec{M}' = 0, \tag{10}$$

which shows that the momentum change is perpendicular to surface of the mirror for any incident angle and mirror orientation. Thus, the momentum change for each reflecting photon lies along the normal to the mirror, independent of mirror speed.

Figure 2 emphasizes the general relationship between P'_b, P'_a, α', and β' when $\Delta\vec{P}'$ is directed along the normal line. Because $\Delta\vec{P}'$ is perpendicular to the mirror, the geometry leads to the general relationship

$$\frac{\sin\beta'}{\sin\alpha'} = \frac{P'_b}{P'_a}. \tag{11}$$

This equation may be used to employ a geometrical analysis to obtain the relationship between β' and α'.

IV. GENERAL RELATIONSHIPS BETWEEN INCIDENT ANGLES AND REFLECTED ANGLES

In this section, we determine equations for α' as a function of α, and β' as a function of β. We begin by taking the dot product of the transformed momentum $\vec{P}'_b$ given by Eq. (5) with the transformed mirror vector $\vec{M}'$ in Eq. (9). Using the geometry shown in Fig. 2, we find

$$\vec{P}'_b\cdot\vec{M}' = P'_{bx}M'_x + P'_{by}M'_y = \left|\vec{P}'_b\right|\left|\vec{M}'\right|\sin\alpha'. \tag{12}$$

After substituting for the vector components and their absolute values from Eqs. (5) and (9), we solve for $\sin\alpha'$ to get

$$\sin\alpha' = \frac{\sin\alpha + (v/c)\cos\phi}{\sqrt{\gamma^2[\sin(\alpha-\phi)+v/c]^2+\cos^2(\alpha-\phi)}\cdot\sqrt{1-(v^2/c^2)\cos^2\phi}}. \tag{13}$$

Using a similar procedure, the functional relationship between β' and β is obtained by taking the dot product of the transformed momentum $\vec{P}'_a$ and the transformed mirror vector $\vec{M}'$, giving

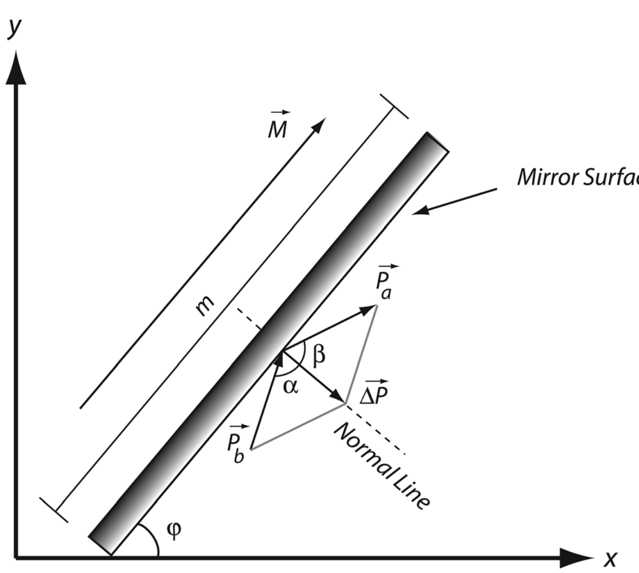

Fig. 1. Light reflecting from a stationary mirror. $\vec{M}$ = mirror vector, m = length of mirror, $\vec{P}_b, \vec{P}_a$ = photon momenta before and after reflection, $\Delta\vec{P} = \vec{P}_a - \vec{P}_b$ = momentum change vector.

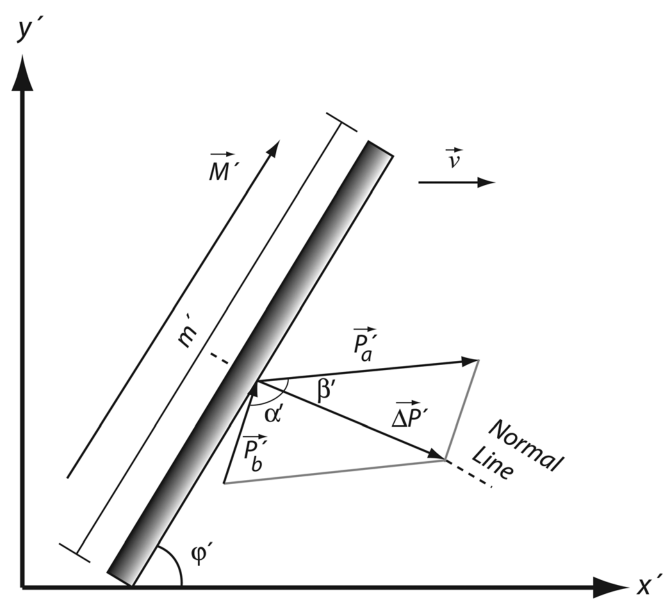

Fig. 2. The experiment shown in Fig. 1 as seen from the $x'y'$-frame. $\vec{M}'$ = mirror vector; m' = length of mirror; $\vec{P}'_b, \vec{P}'_a$ = photon momenta before and after reflection; $\Delta\vec{P}' = \vec{P}'_a - \vec{P}'_b$ = momentum change vector.

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$$\sin\beta' = \frac{\sin\beta + (v/c)\cos\phi}{\sqrt{\gamma^2[\sin(\beta+\phi)+v/c]^2 + \cos^2(\beta+\phi)}\cdot\sqrt{1-(v^2/c^2)\cos^2\phi}}. \tag{14}$$

Note that, in the limit of $v/c \ll 1$, the ratio of Eq. (13) to Eq. (14) becomes

$$\frac{\sin\alpha'}{\sin\beta'} = \frac{\sin\alpha + (v/c)\cos\phi}{\sin\beta + (v/c)\cos\phi}, \tag{15}$$

so that $\alpha' = \beta'$, as expected, when $v/c \ll 1$.

V. EXAMPLES

In Secs. V A and V B, we consider two specific examples where the equations assume much simpler forms.

A. Mirror at rest angle of 45°

As an example, let us apply Eqs. (13) and (14) to the special case where $\phi = 45°$ and also $\alpha = \beta = 45°$ (see Fig. 3). This case is of special interest because a system of mirrors at $\phi = 45°$ can be employed to analyze a "Square Light Clock" to derive time dilation and length contraction independently.[11] Also, a 45° mirror leads to a more intuitive understanding of why the angles of reflection and incidence are not equal when the mirror travels at relativistic speeds. Using these specific angles, Eqs. (13) and (14) reduce to

$$\sin\alpha' = \frac{(1+v/c)\sqrt{1-v^2/c^2}}{\sqrt{2-v^2/c^2}}, \tag{16}$$

$$\sin\beta' = \frac{\sqrt{1-v^2/c^2}}{\sqrt{2-v^2/c^2}}. \tag{17}$$

Taking $v/c = 0.5$, we get $\alpha' = 79°$ and $\beta' = 41°$. The interesting point to notice is that despite the fact that β' is no longer equal to 45° (as in the rest frame), the reflected beam still remains parallel to the x-axis (see Fig. 4). This property is employed in the study of the "Square Light Clock" of Ref. 11. We also note that as $v \to 0$, Eqs. (16) and (17) give $\sin\alpha' = \sin\beta' = 1/\sqrt{2}$. Thus for the mirror at rest we have $\alpha' = \beta' = 45°$ as expected. Finally, combining Eqs. (13) and (14) with Eq. (11), we find for this special case

$$\frac{P'_b}{P'_a} = \frac{\sin\beta'}{\sin\alpha'} = \frac{1}{1+v/c}. \tag{18}$$

This shows that β' is less than α' and also that P'_b is less than P'_a when v/c is greater than zero.

B. Mirror at rest angle of 90°

As another example we consider the case where $\phi = 90°$. Because of its simple geometry, other authors (including Einstein) have studied this special case.[2,3,5,7–10] For $\phi = 90°$, Eqs. (13) and (14) simplify considerably to give

$$\sin\alpha' = \frac{\sin\alpha}{\gamma(1-(v/c)\cos\alpha)}, \tag{19}$$

$$\sin\beta' = \frac{\sin\beta}{\gamma(1+(v/c)\cos\beta)}. \tag{20}$$

As in Subsection V A, we now find a functional relationship between α' and β'. Although carried out for the specific case of $\phi = 90°$, this derivation can be applied to the general case as well.

We begin with Eqs. (1)–(4) when $\phi = 90°$ and calculate $P'_{bx} = \vec{P}'_b \cdot \hat{i}$ to get

$$P'_{bx} = |\vec{P}'_b|\cos\alpha'. \tag{21}$$

By substituting $\phi = 90°$ in Eqs. (1) and (5) and simplifying the relationships for P'_{bx} and P'_b, we can solve Eq. (2) for $\cos\alpha'$ to show that

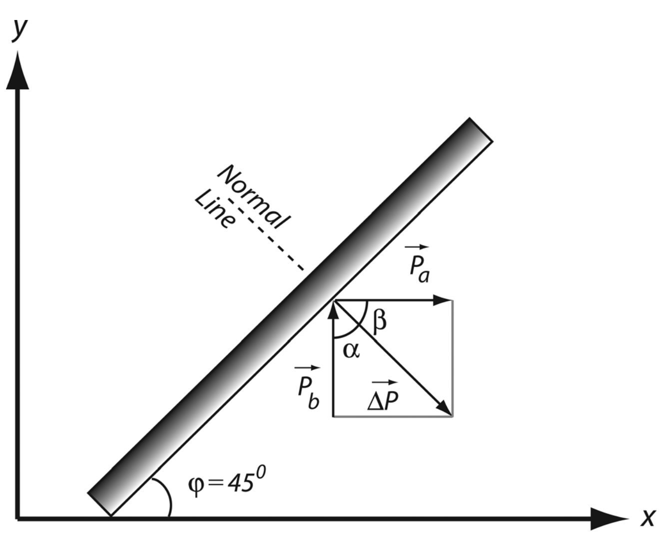

Fig. 3. Stationary mirror at an angle $\phi = 45°$ with $\alpha = \beta = 45°$.

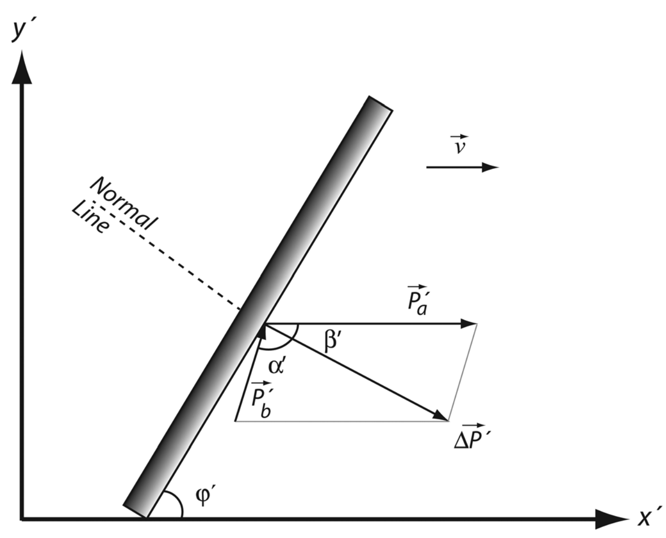

Fig. 4. The experiment shown in Fig. 3 as seen from the x' y'-frame when $v/c = 0.5$. The incident and reflected angles are calculated using Eqs. (16) and (17). (Note that $\vec{P}'_a$ remains horizontal).

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$$\cos\alpha' = \frac{\cos\alpha - v/c}{1-(v/c)\cos\alpha}. \tag{22}$$

Performing the same calculation for $P'_{bx} = \vec{P}'_a \cdot \hat{\imath}$ and solving for $\cos\beta'$ yields

$$\cos\beta' = \frac{\cos\beta + v/c}{1+(v/c)\cos\beta}. \tag{23}$$

Using the fact that $\cos\alpha = \cos\beta$, we can solve for $\cos\alpha$ in Eq. (22) and substitute this in for $\cos\beta$ in Eq. (23) to get

$$\cos\beta' = \frac{(1+v^2/c^2)\cos\alpha' + 2(v/c)}{(1+v^2/c^2) + 2(v/c)\cos\alpha'}. \tag{24}$$

This equation, with the mirror moving in the opposite direction, was derived in Refs. 2, 7, and 8 using different approaches.

We can obtain a different form for the functional relationship between α' and β' by considering the ratio of Eqs. (19) and (20) to get

$$\frac{\sin\alpha'}{\sin\beta'} = \frac{1+(v/c)\cos\beta}{1-(v/c)\cos\alpha}. \tag{25}$$

Replacing $\cos\alpha$ and $\cos\beta$ from Eqs. (22) and (23) results in Ref. 10

$$\frac{\sin\alpha'}{1+(v/c)\cos\alpha'} = \frac{\sin\beta'}{1-(v/c)\cos\beta'}. \tag{26}$$

This equation provides a relationship between the angle of incidence (α') and the angle of reflection (β'), both in the lab frame for the special case of $\phi = 90°$. A more general relationship for these angles is derived in Ref. 7.

VI. CONCLUSIONS AND SUMMARY

In this paper, we have introduced the general principle that the specular reflection of light from a mirror is such that the momentum change of the reflected photons is perpendicular to the plane of the mirror. We have shown that this principle is valid for all mirrors, moving, or stationary. The commonly stated law of reflection—the angle of incidence is equal to the angle of reflection—is valid only for the special case of reflection from mirrors at rest. A more general relationship for reflecting photons from all mirrors is instead given by

$$\frac{\sin\beta'}{\sin\alpha'} = \frac{P'_b}{P'_a}. \tag{27}$$

Because photon momenta are directly proportional to frequency, this equation can also be used to obtain other relationships, such as Doppler frequency ratios.

We also calculated the moving incident angle (α') as a function of the rest incident angle (α), as well as the moving reflected angle (β') as a function of the rest reflected angle (β). In addition, we derived the relationship between β' and α' for two special cases, and demonstrated that this result can be carried out in general. In all cases, the special and the general relationships derived in this paper are in agreement with the results of others who have used different approaches to analyze light reflection from moving mirrors.[2,3,5,7–10]

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