Physics 4610
Quantum Mechanics
Exam 1
Spring Semester 2014

Notes:
You may use your textbook and a table of integrals.

Name: Key

1. A particle, which is confined to an infinite square well of width L, has a wavefunction given by,

$$\psi(x) = \sqrt{\frac{2}{L}} Sin(\frac{2\pi}{L}x)$$

a) Calculate the expectation value of position x and momentum p.

$$\langle x \rangle = \frac{2}{L} \int_0^L x \sin^2 \frac{2\pi}{L} x \, dx = \boxed{\frac{L}{2}}$$

$$\langle p \rangle = \frac{2}{L} \frac{\hbar}{i} \int_0^L \sin \frac{2\pi}{L} x \cdot \underbrace{\frac{d}{dx} \sin \frac{2\pi}{L} x}_{\frac{2\pi}{L} \cos \frac{2\pi}{L} x} \, dx = \cdots$$

$$\boxed{\langle p \rangle = 0}$$

b) Calculate the expectation of energy E.

$$\langle E \rangle = \int \psi^*(x) H \psi(x) dx$$

In this case $\psi(x) = \psi_2(x)$ of infinite square well

$$\Rightarrow H \psi_2(x) = E_2 \psi_2(x) \Rightarrow \langle E \rangle = E_2 \underbrace{\int \psi_2^* \psi_2 dx}_{=1}$$

$$\langle E \rangle = E_2 = \frac{4\pi^2 \hbar^2}{2mL^2}$$

c) Calculate the uncertainty σ_E and explain your results.

$$\langle E^2 \rangle = \int \psi_2^*(x) \underbrace{H^2 \psi(x)}_{E_2^2 \psi_2(x)} dx = E_2^2$$

$$\sigma_E^2 = \langle E^2 \rangle - \langle E \rangle^2 = E_2^2 - E_2^2 = 0$$

The particle is in an stationary state (well defined energy states); Thus $\Delta E = 0$.

2. The state of a particle confined to an infinite square well of width L is given as

$$\psi(x,0) = Nx$$

where N is a normalization constant.

a) Normalize the wave function $\psi(x,0)$.
b) Express this wavefunction as a superposition of the eigenstates of the infinite square well.
c) What is the wavefunction of particle at a later time t?
d) What is the probability of finding the particle with the energy $E = E_2$?

a) $\int |\psi(x,0)|^2 dx = 1 \Rightarrow N^2 \int_0^L x^2 dx = 1$

$N^2 \left[L^3/3 \right] = 1 \rightarrow N = \sqrt{3/L^3} = \frac{1}{L}\sqrt{\frac{3}{L}}$

$$\boxed{\psi(x,0) = \frac{1}{L}\sqrt{\frac{3}{L}}\, x}$$

b) $\psi(x,0) = \sum_n c_n \psi_n(x) \rightarrow c_n = \int \psi(x,0)\, \psi_n(x)\, dx$

but $\psi_n(x) = \sqrt{\frac{2}{L}} \sin\frac{n\pi}{L}x \Rightarrow c_n = \frac{\sqrt{6}}{L^2} \int_0^L x \sin\frac{n\pi}{L}x \, dx$

$$\boxed{c_n = \frac{\sqrt{6}}{n\pi}(-1)^{n+1}} \Rightarrow$$

$$\psi(x,0) = \sum_n \frac{\sqrt{6}}{n\pi}(-1)^{n+1} \cdot \sqrt{\frac{2}{L}} \sin\frac{n\pi}{L}x$$

c) $P(E=E_2) = |c_2|^2 = \left|\frac{\sqrt{6}}{2\pi}\right|^2 = \frac{6}{4\pi^2} = 0.15$

$= 15\%$

3. An electron is acted by a potential $V(x)$ within the region $0 \leq x \leq a$. Its wavefunction is given as

$$\psi(x) = N\sin\left(\frac{\pi x}{a}\right)e^{-i\omega t} \qquad 0 \leq x \leq a$$
$$\psi(x) = 0 \qquad otherwise$$

a) Calculate the normalization constant N.
b) Calculate the probability of finding the electron in the interval $a/4 \leq x \leq 3a/4$.
c) Find the potential $V(x)$.

a) $$\int |\psi(x)|^2 dx = 1 \Rightarrow N^2 \underbrace{\int_0^a \sin^2 \frac{\pi}{a} x \, dx}_{a/2} = 1$$

$$\Rightarrow N = \sqrt{\frac{2}{a}}$$

$$\boxed{\psi(x,t) = \sqrt{\frac{2}{a}} \sin \frac{\pi}{a} x \, e^{-i\omega t}}$$

b) $$P = \int_{a/4}^{3a/4} |\psi(x,t)|^2 dx = \frac{2}{a} \int_{a/4}^{3a/4} \sin^2 \frac{\pi}{a} x \, dx$$

$$= \dots = \frac{1}{2} + \frac{1}{\pi} = 0.82$$

c) $$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x,t) + V\psi(x,t) = i\hbar \frac{\partial}{\partial t} \psi(x,t)$$

$$\frac{\hbar^2}{2m} \cdot \cancel{\sqrt{\frac{2}{a}}} \frac{\pi^2}{a^2} \cancel{\sin \frac{\pi}{a} x \, e^{-i\omega t}} + V \cancel{\sqrt{\frac{2}{a}} \sin \frac{\pi}{a} x \, e^{-i\omega t}} = i\hbar(-i\omega) \cancel{\sqrt{\frac{2}{a}} \sin \frac{\pi}{a} x \, e^{-i\omega t}}$$

$$\Rightarrow \boxed{V = -\frac{\hbar^2 \pi^2}{2ma^2} + \hbar\omega}$$

4. At $t = 0$, a particle in a harmonic-oscillator potential is in the initial state

$$\psi(x,0) = \frac{1}{\sqrt{5}}\psi_1(x) + \frac{2}{\sqrt{5}}\psi_2(x)$$

a) Calculate the expectation value of energy in the state $\psi(x,0)$.
b) Find the state of the particle $\psi(x,t)$ at a later time t. Is this state a stationary state?
c) Calculate the expectation value of x for the state $\psi(x,t)$.
d) What is the frequency of oscillation of this expectation value?

a) $$\langle E\rangle = \int \psi^*(x,0)\, H\, \psi(x,0)\, dx$$

$$= \int \left[\frac{1}{\sqrt{5}}(\psi_1) + \frac{2}{\sqrt{5}}\psi_2\right]\left[\frac{1}{\sqrt{5}}E_1\psi_1 + \frac{2}{\sqrt{5}}E_2\psi_2\right] dx$$

$$\langle E\rangle = \frac{1}{5}E_1 + \frac{4}{5}E_2 = \frac{1}{5}\left(\frac{3\hbar\omega}{2}\right) + \frac{4}{5}\left(\frac{5}{2}\hbar\omega\right) = \boxed{\frac{23}{10}\hbar\omega}$$

b) $$\psi(x,t) = \frac{1}{\sqrt{5}}\psi_1(x)\, e^{-iE_1t/\hbar} + \frac{2}{\sqrt{5}}\psi_2(x)\, e^{-iE_2t/\hbar}$$

This state is NOT an stationary state because its Prob. density depends on time.

c) $$\langle x\rangle = \cancel{\int\left(\frac{1}{\sqrt{5}}\psi_1 + \frac{2}{\sqrt{5}}\psi_2\right) x \left(\frac{1}{\sqrt{5}}\psi_1 + \frac{2}{\sqrt{5}}\psi_2\right)}$$

$$\langle x\rangle = \int\left(\frac{1}{\sqrt{5}}\psi_1 e^{-iE_1t/\hbar} + \frac{2}{\sqrt{5}}\psi_2 e^{-iE_2t/\hbar}\right) x \left(\text{Same thing}\right) dx$$

$$= \frac{1}{5}\underbrace{\cancel{\int x|\psi_1|^2 dx}}_{0} + \frac{4}{5}\underbrace{\cancel{\int x|\psi_2|^2 dx}}_{0} + \frac{2}{5}\int x\psi_1\psi_2 \cdot 2\cos\frac{(E_1-E_2)t}{\hbar}$$

$$\boxed{\langle x\rangle = \frac{2}{5}\cdot 2\cdot\sqrt{\frac{\hbar}{m\omega}}\cos\left(\frac{E_1-E_2}{\hbar}\right)t}$$

$$\omega = \text{frequency} = (E_2 - E_1)/\hbar$$