Physics 4610

Quantum Mechanics

Test 2

Spring Semester 2014

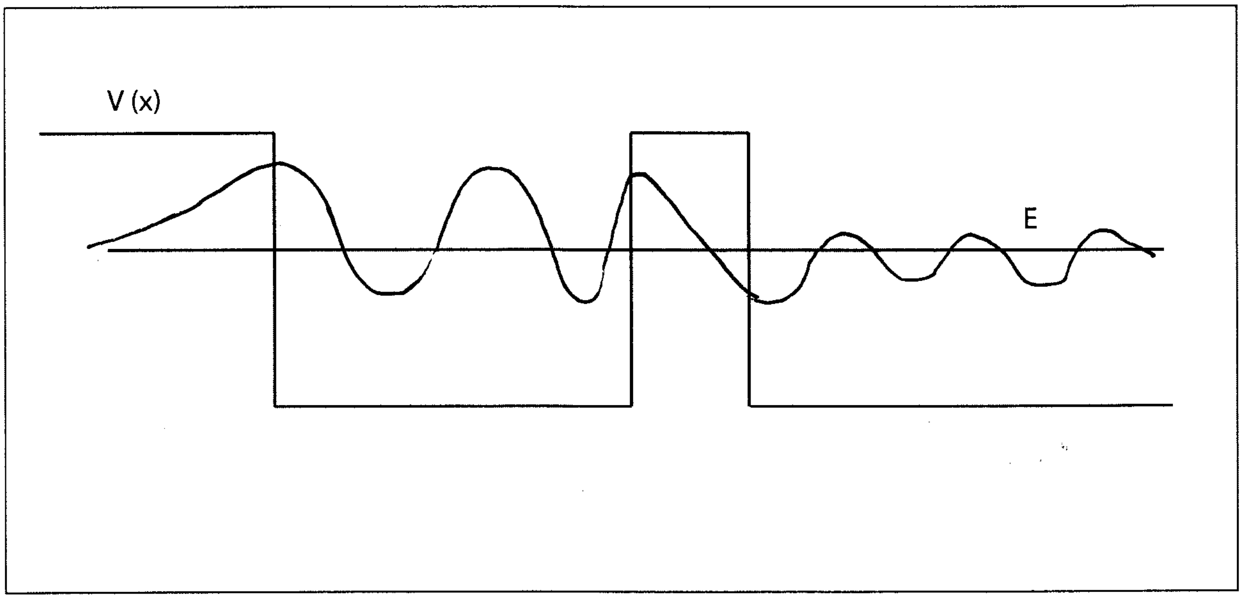
Notes:

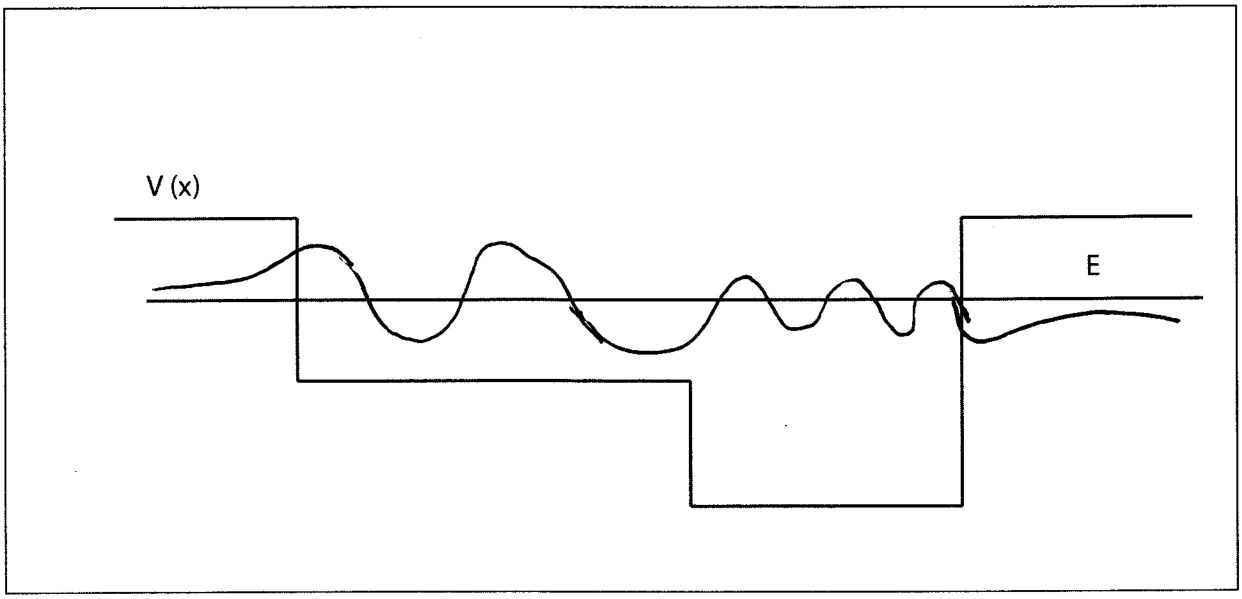
1) The test is open book.
2) You may use a calculator and a table of integrals.

NAME:

Key

1) For the potentials shown below, make a careful sketch of the wavefunctions.





1) If operators $\hat{A}$ and $\hat{B}$are hermitian operators, prove that the operator $\hat{Q} = \hat{A}\hat{B}$ is hermitian if $\hat{A}$ and $\hat{B}$ commute.

$$\langle \psi | Q\psi \rangle = \langle \psi | AB\psi \rangle = \langle A\psi | B\psi \rangle$$

$$= \langle BA\psi | \psi \rangle$$

Thus for $Q = AB$ to be hermitian, we must have

$$AB = BA \quad \text{or} \quad [A, B] = 0$$

3) Consider the operator $\hat{Q} = \hat{x}\hat{p}^2\hat{x}$. Prove that this operator is hermitian.

$$\langle \psi | x p^2 x \psi \rangle = \langle x\psi | p^2 x \psi \rangle$$

$$= \langle p^2 x \psi | x \psi \rangle = \langle x p^2 x \psi | \psi \rangle$$

$\Rightarrow$ Q is hermitian

4) An electron is confined to a potential $V(x) = \frac{1}{2}kx^2$.

a) Calculate the commutation relation $[H, p]$, where H is the Hamiltonian.

a) $[H, p] = \left[\frac{p^2}{2m} + \frac{1}{2}kx^2, p\right] = \frac{1}{2}k[x^2, p]$

but $[x^2, p] = 2i\hbar x$

$$\boxed{[H, p] = i\hbar kx}$$

b) Use the results of part (a) to calculate $\frac{d}{dt} < p >$. Is momentum conserved?

$\frac{d}{dt}\langle p\rangle = \frac{-1}{i\hbar}\langle [H, p]\rangle = \frac{-1}{\cancel{i\hbar}}\langle \cancel{i\hbar} kx\rangle$

$\frac{d}{dt}\langle p\rangle = -k\langle x\rangle$

but $\langle x\rangle = \underbrace{\int_{-\infty}^{+\infty} \psi^* x \psi \, dx}_{\text{odd}} = 0$

$\Rightarrow \frac{d}{dt}\langle p\rangle = 0$

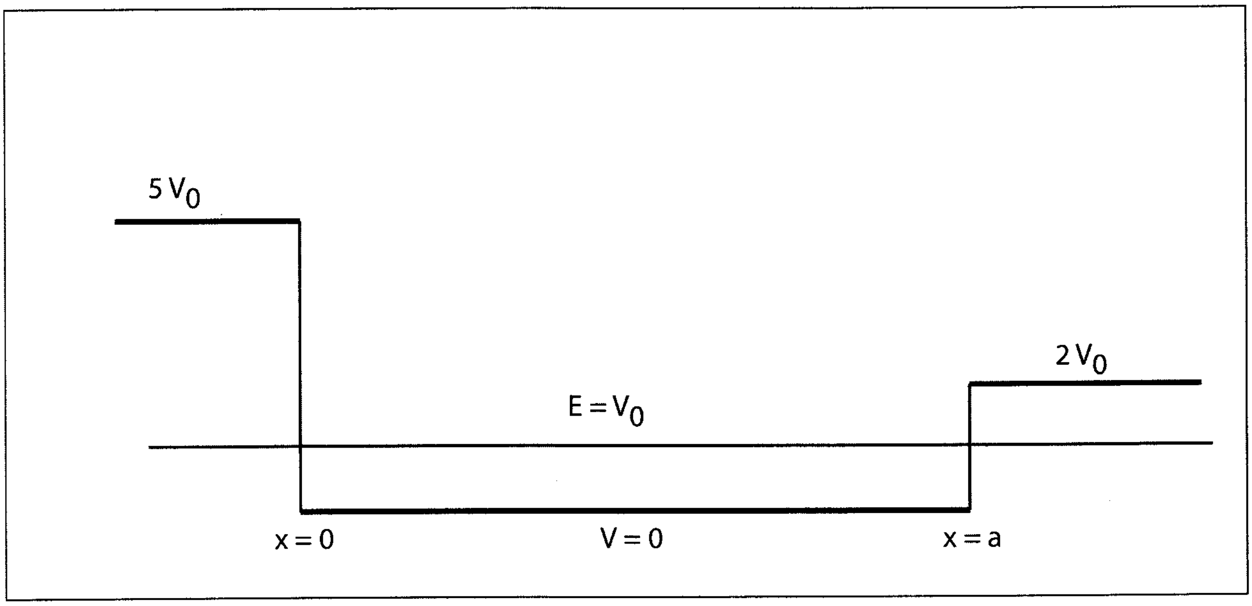
Yes, momentum is conserved.

5) An electron with energy $E = V_0$ is bound to a finite square well potential shown in the figure.

$$V(x) = 5V_0 \quad x \le 0$$
$$V(x) = 0 \quad 0 \le x \le a$$
$$V(x) = 2V_0 \quad x \ge a$$

We define parameter $\alpha^2 = \frac{2mV_0}{\hbar^2}$. All solutions must be expressed in terms of this parameter.

a) Write the wavefunction for each of the three regions (i.e. $x < 0, x > 0$ and $0 < x < a$). Be sure to express the exponents of each wavefunction in terms of the parameter α.
b) Apply the boundary conditions at point $x = 0$.
c) Apply the boundary conditions at point $x = a$. Simplify your results.
d) Use the results you obtained from the boundary conditions to calculate the lowest energy of the system.



a) $x < 0$

$\psi = Ae^{kx}$

$k^2 = \frac{2m}{\hbar^2}(V-E)$

$= \frac{2m}{\hbar^2}(5V_0 - V_0)$

$= \frac{2m}{\hbar^2} 4V_0$

$= 4\alpha^2$

$\Rightarrow \boxed{\psi_{x<0} = Ae^{2\alpha x}}$

$$\psi_{x>0} = B e^{-k'x} \quad \text{where } k'^2 = \frac{2m}{\hbar^2}(V-E) = \frac{2m}{\hbar^2}\underset{V_0}{(\cancel{2V_0 - V_0})}$$
$$= \alpha^2$$

$$\boxed{\psi_{x>0} = B e^{-\alpha x}}$$

$$\psi_{0 \le x \le a} = C \cos k''x + D \sin k''x$$

$$k''^2 = \frac{2m}{\hbar^2}(E-V) = \frac{2mV_0}{\hbar^2} = \alpha^2$$

$$\boxed{\psi_{0 \le x \le a} = C \cos \alpha x + D \sin \alpha x}$$

b) $x = 0 \Rightarrow \begin{cases} A = C \\ 2\cancel{\alpha} A = D\cancel{\alpha} \end{cases} \Rightarrow \boxed{\begin{matrix} A = C \\ D = 2A \end{matrix}}$

c) $x = a \Rightarrow \begin{cases} B e^{-\alpha a} = C \cos \alpha a + D \sin \alpha a \\ -\cancel{\alpha} B e^{-\alpha a} = -C\cancel{\alpha} \sin \alpha a + D \cancel{\alpha} \cos \alpha a \end{cases}$

Divide each side: $-1 = \dfrac{\overset{A}{\cancel{C}} \cos \alpha a + \overset{2A}{\cancel{D}} \sin \alpha a}{-\underset{A}{\cancel{C}} \sin \alpha a + \underset{2A}{\cancel{D}} \cos \alpha a}$

A's cancel and we get

$$\cos\alpha a + 2\sin\alpha a = \sin\alpha a - 2\cos\alpha a$$

$$\Rightarrow \sin\alpha a = -3\cos\alpha a$$

$$\tan\alpha a = -3 \qquad \alpha a = -0.4\pi \quad \text{(radians)}$$

$$\alpha = -\frac{0.4\pi}{a}$$

$$\alpha^2 = \frac{0.16\pi^2}{a^2} = \frac{2m}{\hbar^2} V_0 \Rightarrow$$

$$\boxed{V_0 = \frac{0.16\pi^2\hbar^2}{2ma^2} = E}$$