2.10

a) $\psi_2(x) = ?$

$$\psi_{n+1} = \frac{a_+}{\sqrt{n+1}}\psi_n \Rightarrow \text{ for } n=1 \text{ we get:}$$

$$\psi_2 = \frac{a_+}{\sqrt{1+1}}\psi_1 = \frac{a_+}{\sqrt{2}}\psi_1$$

but $\psi_1 = \frac{a_+}{\sqrt{0+1}}\psi_0 = a_+\psi_0$

$$\Rightarrow \psi_1 = \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega x - \hbar\frac{d}{dx}\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\Rightarrow \psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\, x\, e^{-\frac{m\omega}{2\hbar}x^2}$$

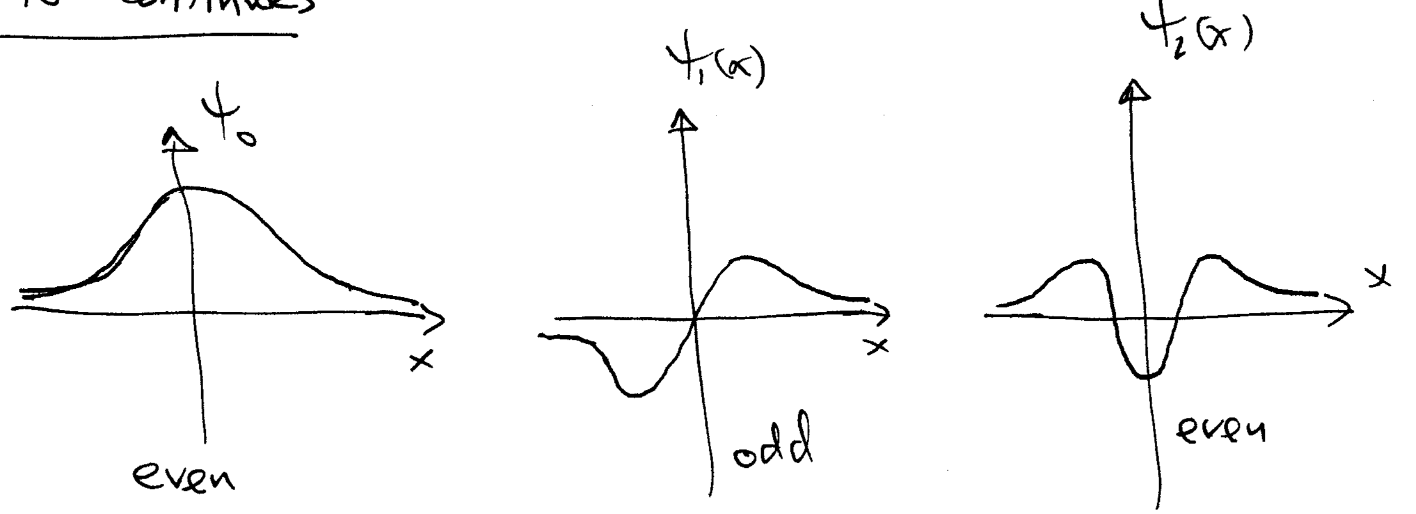
and

$$\psi_2 = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega x - \hbar\frac{d}{dx}\right)\psi_1(x)$$

$$= \dots$$

$$\boxed{\psi_2(x) = \frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)e^{-\frac{m\omega}{2\hbar}x^2}}$$

2.10 Continues



c) Orthogonality:

$$\int_{-\infty}^{+\infty} \underbrace{\psi_0 \psi_1}_{\text{odd function}} \, dx = 0$$

$$\int_{-\infty}^{+\infty} \underbrace{\psi_2 \psi_1}_{\text{odd function}} \, dx = 0$$

$$\int_{-\infty}^{+\infty} \psi_0 \psi_2 \, dx = (\text{constants}) \int_{-\infty}^{+\infty} e^{-\frac{m\omega}{2\hbar} x^2} \cdot \left(\frac{2m\omega}{\hbar} x^2 - 1 \right) e^{-\frac{m\omega}{2\hbar} x^2}$$

$$= \dots = 0$$

2.12

$$\langle x \rangle = \int \psi_n^* \, x \, \psi_n \, dx$$

$$x \to \sqrt{\frac{\hbar}{2m\omega}} \left(a_+ + a_-\right)$$

but $a_+ \psi_n = \sqrt{n+1}\, \psi_{n+1}$

and $\int \psi_n \psi_{n+1} \, dx = 0$

Similarly: $a_- \psi_n = \sqrt{n}\, \psi_{n-1}$ and

$\int \psi_n \psi_{n-1} \, dx = 0 \longrightarrow$

$$\boxed{\langle x \rangle = 0}$$

$$\langle p \rangle = \int \psi_n^* \, p \, \psi_n \, dx = \int \psi_n \, i \sqrt{\frac{\hbar m\omega}{2}} \left(a_+ - a_-\right) \psi_n$$

$= 0$ (similar to $\langle x \rangle$)

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega} \int \psi_n \Big(\underbrace{a_+^2 + a_-^2}_{\text{do not contribute}} + \underbrace{a_+ a_- + a_- a_+}_{= \frac{2H}{\hbar\omega}} \Big) \psi_n$$

2.12 Continues

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\cdot\frac{2}{\hbar\omega}\int \psi_n \underbrace{H\psi_n}_{E_n\psi_n} dx$$

$$\downarrow$$

$$E_n = (n+\tfrac{1}{2})\hbar\omega$$

$$\Rightarrow \langle x^2\rangle = \frac{\hbar}{\cancel{2}m\omega}\cdot\frac{\cancel{2}}{\cancel{\hbar\omega}}\cdot(n+\tfrac{1}{2})\cancel{\hbar\omega}$$

$$\boxed{\langle x^2\rangle = (n+\tfrac{1}{2})\frac{\hbar}{m\omega}}$$

$$\langle p^2\rangle = \int \psi_n\left(-\frac{\hbar m\omega}{2}\left(a_+^2+a_-^2-a_-a_++a_+a_-\right)\right)\psi_n\, dx$$

Similar arguments as $\langle x^2\rangle$ yields

$$\boxed{\langle p^2\rangle = (n+\tfrac{1}{2})m\hbar\omega}$$

$$\langle T\rangle = \left\langle \frac{p^2}{2m}\right\rangle = \frac{1}{2}(n+\tfrac{1}{2})\hbar\omega$$

$$\left.\begin{aligned}\sigma_x &= \sqrt{\langle x^2\rangle - \cancel{\langle x\rangle}^2} = \sqrt{\frac{\hbar}{m\omega}(n+\tfrac{1}{2})}\\ \sigma_p &= \sqrt{\langle p^2\rangle - \cancel{\langle p\rangle}^2} = \sqrt{m\hbar\omega(n+\tfrac{1}{2})}\end{aligned}\right\} \Rightarrow \sigma_x\sigma_p = (n+\tfrac{1}{2})\hbar \geq \hbar/2$$

2.13

$$\Psi(x,0) = A\left[3\psi_0(x) + 4\psi_1(x)\right]$$

a) $\int |\Psi(x,0)|^2 \, dx = 1 \Rightarrow$

$$A^2 \int \left(3\psi_0(x) + 4\psi_1(x)\right)\left(3\psi_0(x) + 4\psi_1(x)\right) dx = 1$$

$$\Rightarrow A^2\left[9 + 16\right] = 1 \rightarrow A = \frac{1}{\sqrt{25}} = \frac{1}{5}$$

$$\boxed{\Psi(x,0) = \frac{3}{5}\psi_0(x) + \frac{4}{5}\psi_1(x)}$$

b) $$\Psi(x,t) = \frac{3}{5}\psi_0(x)\, e^{\frac{-iE_0 t}{\hbar}} + \frac{4}{5}\psi_1(x)\, e^{-iE_1 t/\hbar}$$

where $E_0 = \hbar\omega/2$ $\quad E_1 = \frac{3}{2}\hbar\omega$

$$|\Psi(x,t)|^2 = \frac{9}{25}|\psi_0(x)|^2 + \frac{16}{25}|\psi_1(x)|^2 +$$

$$\frac{12}{25}\psi_0(x)\psi_1(x) \cdot 2\cos\left(\frac{E_1 - E_0}{\hbar}t\right)$$

2.13 Continues

$$\langle x \rangle = \int \Psi^*(x,t)\, x\, \Psi(x,t)\, dx$$

$$= \int x\, |\Psi(x,t)|^2\, dx$$

Terms like $\int x |\psi_0(x)|^2 dx$ and $\int x |\psi_1(x)|^2 dx$

do not contribute and they vanish.

Thus we need to evalute:

$$\int_{-\infty}^{+\infty} x\, \psi_0(x)\, \psi_1(x)\, dx = \cdot\cdot = \sqrt{\frac{\hbar}{2m\omega}}$$

$$\Rightarrow \langle x \rangle = \frac{24}{25} \sqrt{\frac{\hbar}{2m\omega}} \cos \omega t$$

$$\text{where } \omega = \frac{E_1 - E_0}{\hbar} t$$

$$\langle p \rangle = m \frac{d\langle x \rangle}{dt} = -\frac{24}{25} \sqrt{\frac{m\hbar\omega}{2}} \sin \omega t$$

d) we get $E_0 = \frac{\hbar\omega}{2}$ with Prob. $= 9/25$

and $E_1 = \frac{3}{2}\hbar\omega$ with Prob. $= 16/25$