1.7

$$\frac{d\langle p\rangle}{dt} = ?$$

$$\langle p\rangle = \int \Psi^* \frac{\hbar}{i}\frac{\partial}{\partial x}\Psi \, dx = \frac{\hbar}{i}\int \Psi^* \frac{\partial \Psi}{\partial x} dx$$

$$\frac{d\langle p\rangle}{dt} = \frac{\hbar}{i}\left[\int \frac{\partial \Psi^*}{\partial t}\frac{\partial \Psi}{dx} dx + \underbrace{\int \Psi^* \frac{\partial}{\partial x}\frac{\partial \Psi}{\partial t} dx}_{= -\int \frac{\partial \Psi^*}{\partial x}\frac{\partial \Psi}{\partial t} dx}\right]$$

from Schrödinger's Eq. $\Rightarrow$

$$-\frac{\hbar}{i}\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$$

$$\Rightarrow \begin{cases} \frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\frac{-i}{\hbar}\frac{\partial^2 \Psi}{\partial x^2} \cancel{+} - \frac{i}{\hbar}V\Psi \\ \frac{\partial}{\partial t}\Psi^* = -\frac{\hbar^2}{2m}\frac{+i}{\hbar}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^* \end{cases}$$

$$\frac{d\langle p\rangle}{dt} = \frac{\hbar}{i}\left[\int\left(-\frac{\hbar^2}{2m}\frac{+i}{\hbar}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*\right)\frac{\partial \Psi}{\partial x} dx \right.$$

$$\left. - \int\left(-\frac{\hbar^2}{2m}\frac{-i}{\hbar}\frac{\partial^2 \Psi}{\partial x^2} \cancel{+} - \frac{i}{\hbar}V\Psi\right)\frac{\partial \Psi^*}{\partial x} dx\right]$$

1.7 Continues

Let's check the first term in each integral:

$$\frac{-i}{\hbar}\frac{\hbar^2}{2m}\int \frac{\partial^2\psi^*}{\partial x^2}\frac{\partial\psi}{\partial x}\,dx - \frac{i}{\hbar}\frac{\hbar^2}{2m}\int\frac{\partial^2\psi}{\partial x^2}\frac{\partial\psi^*}{\partial x}\,dx$$

$$= -\frac{\hbar^2}{2m}\frac{i}{\hbar}\left[\underbrace{\int \frac{\partial^2\psi^*}{\partial x^2}\frac{\partial\psi}{\partial x}\,dx}_{\downarrow} + \int\frac{\partial^2\psi}{\partial x^2}\frac{\partial\psi^*}{\partial x}\,dx\right]$$

$$\frac{\partial}{\partial x}\left(\frac{\partial\psi^*}{\partial x}\frac{\partial\psi}{\partial x}\right) - \frac{\partial\psi^*}{\partial x}\frac{\partial^2\psi}{\partial x^2}$$

The 2nd term cancels the second term in the bracket. The first term can be integrated and goes to zero at the limits.

Now let's check the 2nd terms involving V:

$$\frac{d\langle p\rangle}{dt} = \frac{\hbar}{i}\cdot\frac{+i}{\hbar}\int V\psi^*\frac{\partial\psi}{\partial x}\,dx + \frac{\hbar}{i}\frac{i}{\hbar}\int V\psi\frac{\partial\psi^*}{\partial x}\,dx$$

1.7 Continues

$$\frac{d\langle p\rangle}{dt} = \int V\psi^* \frac{\partial \psi}{\partial x}\,dx + \underbrace{\int V\psi \frac{\partial \psi^*}{\partial x}\,dx}$$

$\downarrow$

integration by parts =>

$$\cancel{\psi^* V\psi}\Big|_{-\infty}^{+\infty} - \int \psi^* \frac{\partial}{\partial x}(V\psi)\,dx$$

0

$$= -\int \psi^* \frac{\partial V}{\partial x}\psi\,dx \; \underbrace{- \int \psi^* V \frac{\partial \psi}{\partial x}}$$

$\downarrow$

cancels the first term =>

$$\frac{d\langle p\rangle}{dt} = -\int \psi^* \frac{\partial V}{\partial x}\psi\,dx$$

$$= -\left\langle \frac{\partial V}{\partial x} \right\rangle$$

1.9

$$\Psi(x,t) = A e^{-a\left[\frac{mx^2}{\hbar} + it\right]}$$

a) $$\int \Psi^*(x,t) \cdot \Psi(x,t)\, dx = 1$$

$$A^2 \int_{-\infty}^{+\infty} e^{-2amx^2/\hbar}\, dx = 1$$

This is a Gaussian integral :

$$A^2 (2) \int_0^{\infty} e^{-\frac{2am}{\hbar} x^2}\, dx = 1$$

$$\cancel{2} A^2 \left(\frac{\hbar}{2ma}\right)^{1/2} \frac{1}{\cancel{2}} = 1$$

$$\boxed{A = \left(\frac{2ma}{\hbar}\right)^{1/4}}$$

$$\Psi(x,t) = \left(\frac{2ma}{\hbar}\right)^{1/4} e^{-\frac{ma}{\hbar}x^2} \cdot e^{-iat}$$

1.9 Continues

b) Using the Schrodinger's Eq. $\Rightarrow$

$$i\hbar \frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$$

$$\Downarrow$$

$$i\hbar(-ai)\Psi = -\frac{\hbar^2}{2m}\left(-\frac{2ma}{\hbar}\right)\left(1-\frac{2ma}{\hbar}x^2\right)\Psi + V\Psi$$

$$\Rightarrow \cancel{(\hbar a)\Psi} = \cancel{\frac{\hbar^2}{2m}\frac{2ma}{\hbar}\Psi} + \frac{\hbar^2}{2m}\frac{2ma}{\hbar}\left(-\frac{2ma}{\hbar}x^2\right)\Psi + V\Psi$$

$$\Rightarrow \boxed{V = 2ma^2x^2} \quad \checkmark$$

c) $$\langle x \rangle = \int_{-\infty}^{\infty} \underbrace{x|\psi|^2}_{\text{odd functi}} dx = 0$$

$$\langle x^2 \rangle = \int x^2 |\psi|^2 dx = \cdots = \boxed{\frac{\hbar}{4am}}$$

$$\langle p \rangle = m \frac{d}{dt} \langle x \rangle = 0$$

$$\langle p^2 \rangle = \int \psi^* \left(\frac{\hbar}{i} \frac{\partial}{\partial x} \right)^2 \psi \, dx$$

$$= -\hbar^2 \int \psi^* \frac{\partial^2 \psi}{\partial x^2} dx = \cdots = \boxed{am\hbar}$$

d) $$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 \Rightarrow \sigma_x = \sqrt{\frac{\hbar}{4am}}$$

$$\sigma_p^2 = \langle p^2 \rangle - \langle p \rangle^2 \Rightarrow \sigma_p = \sqrt{am\hbar}$$

$$\sigma_x \sigma_p = \sqrt{\frac{\hbar}{4am}} \sqrt{am\hbar} = \boxed{\hbar/2}$$

1.17

$$\psi(x,0) = \begin{cases} A(a^2-x^2) & -a \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

a) $\int |\psi(x,0)|^2 dx = 1 \Rightarrow$

$$A^2 \int_{-a}^{+a} (a^2-x^2)^2 dx = 1$$

$$2A^2 \int_0^a (a^2-x^2)^2 dx = 1 \Rightarrow \boxed{A = \sqrt{\frac{15}{16a^5}}}$$

b) $\langle x \rangle = A^2 \int_{-a}^{+a} \underbrace{x(a^2-x^2)\,dx}_{\text{odd}} = \boxed{0}$

c) $\langle p \rangle = \frac{\hbar}{i} A^2 \int \underbrace{(a^2-x^2) \underbrace{\frac{d}{dx}(a^2-x^2)}_{-2x}\,dx}_{\text{odd}}$

$$\boxed{\langle p \rangle = 0}$$

1.17 Continues

d) $\langle x^2 \rangle = 2A^2 \int_0^a x^2 (a^2 - x^2)^2 \, dx = \cdots$

$$\boxed{\langle x^2 \rangle = \frac{a^2}{7}}$$

e) $\langle p^2 \rangle = -A^2 \hbar^2 \int_{-a}^{a} (a^2 - x^2) \underbrace{\frac{d^2}{dx^2} (a^2 - x^2)}_{-2} \, dx$

$$\boxed{\langle p^2 \rangle = \frac{5 \hbar^2}{2a^2}}$$

f) $\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{a}{\sqrt{7}}$

g) $\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{5/2} \, \hbar/a$

h) $\sigma_x \sigma_p = \sqrt{\frac{10}{7}} \, \hbar/2 > \hbar/2$ ✓