1. **Constructing vector derivatives and integrals.** The accompanying worksheet shows a contour plot of a scalar function $g$ in two dimensions. This function measures a quantity that I’ll call *globbiness* (sorry), measured in units called *globbs*. Each contour denotes a line of constant globbiness, successive contours differ by one glob, and the numerical value in globbs is written on every fifth contour. To work this problem you will need a sharp pencil and a ruler that measures centimeters. A plastic triangle for constructing perpendiculars is helpful but not absolutely necessary.

(a) Directly on the worksheet, carefully draw an arrow to represent the vector field $\mathbf{v} \equiv \nabla g$ at each of the twelve grid points labeled by lower-case letters. To do this you will need to measure the distance between nearby contour lines and do a short calculation for each location. It is necessary to choose a convention for the lengths of the arrows you draw; let the convention be that an arrow of length 1 cm represents a field strength of 1 glob/cm, with stronger and weaker field strengths represented by proportionally longer and shorter arrows.

(b) Measure the $x$ and $y$ components of each of the arrows you constructed in part (a), and record their values in the table on the worksheet. (Try to estimate each component to the nearest half-millimeter on your ruler; there’s no need to try to be more accurate than that.) Use the values in this table for all subsequent operations on $\mathbf{v}$.

(c) Compute the line integral of $\mathbf{v} \cdot d\mathbf{l}$ along the path $DBA$, by breaking the path into four equal segments and approximating the value of $\mathbf{v}$ along each segment to be constant, equal to the value at the center of the segment (which is recorded in your table). Do the same for the line integral along path $DCA$. Also, directly from the contour plot, estimate the values $g(A)$ and $g(D)$. The fundamental theorem for gradients says that

$$\int_{D}^{A} \mathbf{v} \cdot d\mathbf{l} = g(A) - g(D)$$

(and therefore that this line integral is path independent). Do your computations verify the theorem? (To answer this last question you’ll need to briefly discuss the amount of uncertainty in your measurements.)

(d) In two dimensions, the curl of a vector function is a number, defined as

$$\nabla \times \mathbf{v} \equiv \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}.$$ 

Estimate $\nabla \times \mathbf{v}$ at each of the four interior points $W$, $X$, $Y$, and $Z$, in each case using the values of $\mathbf{v}$ at the four surrounding points to estimate the partial derivatives. For instance,

$$\frac{\partial v_y(W)}{\partial x} \approx \frac{v_y(d) - v_y(c)}{\Delta},$$
where $\Delta = 4\,\text{cm}$ is the distance between $c$ and $d$. Add up your four values for the curl and multiply by the appropriate area to obtain an estimate of the integral $\int (\nabla \times \mathbf{v})\,da$ over the entire square area shown. Also estimate the circulation $\oint \mathbf{v} \cdot d\mathbf{l}$ for the closed path ACDB. The fundamental theorem for curls states that

$$\int (\nabla \times \mathbf{v})\,da = \oint \mathbf{v} \cdot d\mathbf{l}.$$

Do your calculations verify this theorem? (They should, exactly.) Write a short paragraph explaining why this “fundamental” theorem is in fact trivial.

(e) Estimate $\nabla \cdot \mathbf{v}$ at each of the interior points $W$, $X$, $Y$, and $Z$, using a method analogous to the way you calculated curls in part (d). Combine your results to find $\int (\nabla \cdot \mathbf{v})\,d\tau$ for the two-dimensional “volume” (really an area) enclosed by the entire square. Also estimate the flux $\oint \mathbf{v} \cdot d\mathbf{a}$ for the closed “surface” (really a line) bounding the square, breaking the surface into eight equal segments and approximating $\mathbf{v}$ along each segment by its value at the center (a grid point). The fundamental theorem for divergences states that

$$\int (\nabla \cdot \mathbf{v})\,d\tau = \oint \mathbf{v} \cdot d\mathbf{a}.$$

Do your calculations verify this theorem? (They should, exactly.) Briefly explain why this “fundamental” theorem is in fact trivial.

2. Prove that the divergence of a curl is always zero. Then check this theorem for the function $x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z}$.

3. Starting with a clear picture or two, derive the formulas for $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ in terms of $\hat{x}$, $\hat{y}$, and $\hat{z}$.

4. (a) Check the divergence theorem for the function $r^2\hat{r}$, using as your volume a sphere of radius $R$, centered at the origin. (Compute the divergence from scratch, using rectangular coordinates.) (b) Do the same for the function $(1/r^2)\hat{r}$. (In this case we’ll compute the divergence in class, and you may simply quote that result.)

5. (a) Use a three-dimensional delta function to write down a formula for the volume charge density, $\rho(\mathbf{r})$, of a point charge $q$ located at $\mathbf{r} = \mathbf{r}'$. Make sure that the volume integral of $\rho$ equals $q$. (b) Use a one-dimensional delta function to write down a formula (in spherical coordinates) for the volume charge density of an infinitely thin spherical shell of charge, centered at the origin, with radius $R$ and total charge $Q$, smeared uniformly over the shell. Make sure that the volume integral of $\rho$ equals $Q$.

6. For “Theorem 1” in Griffiths’s section on the Theory of Vector Fields, show that (d) $\implies$ (a), (a) $\implies$ (c), (c) $\implies$ (b), (b) $\implies$ (c), and (c) $\implies$ (a).

7. Complete the accompanying exercise in LATEX typesetting, on the subject of poisonous snakes.
Vector Derivatives and Integrals Worksheet
(Must be printed at full size, so the large square below measures 8 cm across)

Components of $\mathbf{v}$ (globs/cm):

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<th>point</th>
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<th>$y$ component</th>
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<tbody>
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\LaTeX{} typesetting exercise: Poisonous snakes!

The purpose of this exercise is to give you more practice using the \LaTeX{} typesetting system—specifically, to practice typesetting mathematical symbols and expressions. The exercise is simply to typeset the long paragraph about the “poisonous snake” on page 39 in your textbook (Griffiths, 4th edition).

You’ll encounter quite a few new tasks as you typeset this paragraph, so please read the following tips first:

- All mathematical symbols and expressions should be typeset in “math mode,” which you do by putting a dollar sign ($$) at the beginning and end. So, for example, a symbol like $x$ is typeset as $$x$$.
- To get Greek letters you type out (in math mode!) their names, preceded by the backslash, e.g., $\theta$.
- To put a math symbol in boldface you use $\mathbf{\text{symbol}}$ for ordinary letters and numerals, and $\boldsymbol{\text{symbol}}$ for Greek letters and other symbols. Examples: $\mathbf{A}$, $\boldsymbol{\theta}$. Also, to get $\boldsymbol{\text{symbol}}$ to work you need to put the instruction $\text{usepackage{amsmath}}$ at the top of your source file, right after the $\text{documentclass}$ line.
- To put a hat over a math symbol, you say something like $\hat{\mathbf{x}}$.
- That’s a lot of typing, so if you’re using a bold symbol or symbol combination repeatedly, it’s best to define an abbreviation for it at the beginning of your document. For instance, $\text{newcommand}{\rhat}{\hat{\mathbf{r}}}$ allows you to typeset $\hat{\mathbf{r}}$ by simply typing $\rhat$ (in math mode).
- To put ordinary text in italics, use $\textit{the text}$.
- To put two dots over a letter, put $"$ in front of it.
- To make pretty left curly quote marks, use the back-quote key at the upper-left corner of your keyboard. Hit it twice for double quotes.
- \LaTeX{} leaves extra space after a period, but if a period isn’t the end of a sentence, you can override this behavior by typing either backslash-space or, for an “unbreakable” space, the tilde ($\sim$) symbol.

I hope these are all the tips you need for this exercise, but if something goes wrong, please ask me or a classmate, or try googling your question. Remember that you must do all the typing yourself—no copy/paste from your classmates’ work! Please print your typeset output and turn it in with the rest of your homework.