Very Large Numbers (from Section 2.4)

There are three kinds of numbers that commonly occur in statistical mechanics: small numbers, large numbers, and very large numbers.

Small numbers are small numbers, like 6, 23, and 42. You already know how to manipulate small numbers.

Large numbers are much larger than small numbers, and are frequently made by exponentiating small numbers. The most important large number in statistical mechanics is Avogadro's number, which is of order 10^{23}. The most important property of large numbers is that you can *add* a small number to a large number without changing it. For example,

$$10^{23} + 23 = 10^{23}. \tag{2.12}$$

(The only exception to this rule is when you plan to eventually subtract off the same large number: $10^{23} + 42 - 10^{23} = 42$.)

Very large numbers are even larger than large numbers, and can be made by exponentiating large numbers. An example would be* $10^{10^{23}}$. Very large numbers have the amazing property that you can *multiply* them by large numbers without changing them. For instance,

$$10^{10^{23}} \times 10^{23} = 10^{(10^{23}+23)} = 10^{10^{23}}, \tag{2.13}$$

by virtue of equation 2.12. This property takes some getting used to, but can be extremely convenient when manipulating very large numbers. (Again, there is an exception: When you plan to eventually *divide* by the same very large number, you need to keep track of any leftover factors.)

One common trick for manipulating very large numbers is to take the logarithm. This operation turns a *very* large number into an ordinary *large* number, which is much more familiar and can be manipulated more straightforwardly. Then at the end you can exponentiate to get back the very large number. I'll use this trick later in this section.

Problem 2.12. The natural logarithm function, ln, is defined so that $e^{\ln x} = x$ for any positive number x.

(a) Sketch a graph of the natural logarithm function.

(b) Prove the identities

$$\ln ab = \ln a + \ln b \qquad \text{and} \qquad \ln a^b = b \ln a.$$

(c) Prove that $\dfrac{d}{dx} \ln x = \dfrac{1}{x}$.

(d) Derive the useful approximation

$$\ln(1+x) \approx x,$$

which is valid when $|x| \ll 1$. Use a calculator to check the accuracy of this approximation for $x = 0.1$ and $x = 0.01$.

*Note that x^{y^z} means $x^{(y^z)}$, not $(x^y)^z$.