Problem Set 8
(due Friday, March 18, 4:00 pm)

1. Prove the following commutator identities:
   (a) \([A + B, C] = [A, C] + [B, C]\)
   (b) \([AB, C] = A[B, C] + [A, C]B\)

2. In this problem you will explore some commutation relations involving angular momentum operators.
   (a) Find all of the commutation relations between position and momentum operators in three dimensions, e.g., \([x, y]\), \([p_x, p_y]\), \([x, p_x]\), \([x, p_y]\), etc.
   (b) Find all the commutation relations between \(L_z\) and the position and momentum operators \(x, y, z, p_x, p_y,\) and \(p_z\).
   (c) Use the results of part (b) to evaluate \([L_z, L_x]\).
   (d) Evaluate the commutators \([L_z, r^2]\) and \([L_z, |\vec{p}|^2]\).
   (e) Prove that if the potential energy \(V\) depends only on \(r\), then the Hamiltonian commutes with all three components of \(\vec{L}\). Discuss the implications, in terms of eigenstates and measurements.

3. Use the explicit formulas for \(L_+\) and \(L_-\) in terms of \(\theta\) and \(\phi\) derivatives to show that
   \[
   L_+L_- = -\hbar^2 \left( \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \cot^2 \theta \frac{\partial^2}{\partial \phi^2} + i \frac{\partial}{\partial \phi} \right).
   \]
   (Hint: Use a test function; otherwise you’re likely to drop some terms.) Then use this expression and the formula for \(L^2\) in terms of \(L_+L_-\) and \(L_z\) to express \(L^2\) as a differential operator in spherical coordinates. (The answer is in Griffiths.)

4. In this problem you will use the raising and lowering operators to derive some of the formulas for spherical harmonics.
   (a) What is \(L_+Y^l_l\)? (No calculation allowed!)
   (b) Write the result of part (a) using the explicit form of \(L_+\) in terms of \(\theta\) and \(\phi\) derivatives. Then, assuming that \(Y^l_l(\theta, \phi)\) is separable and using what you know about \(L_+Y^l_l\), obtain a differential equation for the \(\theta\) dependence of \(Y^l_l\) and solve this differential equation to obtain the general formula for \(Y^l_l\). Don’t worry about the normalization constant. Check your formula against the particular tabulated formulas for \(l \leq 3\).
   (c) Now apply \(L_-\) to your formula for \(Y^l_l\) to obtain a general formula for \(Y^{l-1}_l\), and check this against the particular tabulated formulas for \(l \leq 3\). (You could, of course, continue applying \(L_-\) to keep working your way down, but by now you’ve got the general idea.)
5. The best way to visualize the spherical harmonics is as “density plots” on the surface of a sphere, with phases represented by colors. (One disadvantage to this representation is that you can see only half of the function at a time, on the visible side of the sphere.) Working from the tabulated formulas and using your handy colored pencils, sketch plots of all of the \( l = 1 \) and \( l = 2 \) spherical harmonics on the blank spheres provided below (or copies thereof). Take the z axis to point up in all cases, and choose any orientation you like for x and y in the horizontal plane. Use light or heavy shading to qualitatively represent the magnitude of the function, leaving the page white wherever the function has a node.

6. Imagine a diatomic molecule made of two different atoms, such as CO or NO or HCl. In this problem you will explore the rotational states of this molecule, neglecting any translational motion, vibrational excitations, or electronic excitations. Then the configuration of the molecule can be described entirely by the direction of a vector drawn from the center of one atom to the center of the other, and this direction can be specified by the usual spherical coordinates \( \theta \) and \( \phi \). In quantum mechanics, therefore, the wavefunction of this system is a function of these two angles (and no other variables). Please assume that the molecule’s moment of inertia \( I \), about its center of mass, is a given constant.

(a) Recall (or look up) the classical formula for the rotational kinetic energy of a rigid object in terms of its angular momentum and moment of inertia. Use this formula to write down the Hamiltonian operator of this system. (There is no potential energy.)

(b) Explain why the energy eigenfunctions of this system are the spherical harmonics, \( Y_\ell^m(\theta, \phi) \). What are the corresponding energy eigenvalues? Draw an energy level diagram, with a linear vertical scale, showing the lowest four energy levels and their degeneracies.

(c) For carbon monoxide (CO), the difference in energy between the rotational ground state and the first excited level is approximately 0.00048 eV. What frequency of electromagnetic radiation should you use to induce a transition from the ground state to the first excited level? What frequency should you use to induce a transition from the first to the second excited level? What part of the electromagnetic spectrum are we talking about here? Extra credit: Use this measured energy difference to find the approximate distance between the C and O nuclei.