1. Starting with the harmonic oscillator ground-state wavefunction $\psi_0(x)$, apply the raising operator repeatedly to find the first three excited states, $\psi_1(x), \psi_2(x)$, and $\psi_3(x)$. Feel free to use natural units, and don’t worry about normalization.

2. The calculations in your reading handout showed that $a^+ \psi_n = c_+ \psi_{n+1}$ for some constant $c_+$, and similarly $a^- \psi_n = c_- \psi_{n-1}$ for some constant $c_-$. In this problem you will determine the constants $c_+$ and $c_-$.  

(a) First prove that $a^-$ and $a^+$ are adjoints of each other: $a^\dagger_- = a^+$ and $a^\dagger_+ = a^-$. (You may assume that the operators $x$ and $p$ are Hermitian.)

(b) Now evaluate the inner product $\langle \psi_n, H \psi_n \rangle$ in three different ways: first by letting $H$ act directly on $\psi_n$, then by expressing $H$ in terms of $a^+ a^-$, and finally by expressing $H$ in terms of $a^- a^+$. In each of these latter cases you can move the leftmost ladder operator to the other side of the inner product by converting it to its adjoint. Then compare the three results to determine $c_+$ and $c_-$. There is an undetermined phase factor $e^{i\phi}$, but you can set it to 1 (that is, take $c_+$ and $c_-$ to be real and positive) without harm, because the phase factor doesn’t affect the normalization of $\psi_n$. You should end up with the relations

$$a^- \psi_n = \sqrt{n} \psi_{n-1}, \quad a^+ \psi_n = \sqrt{n+1} \psi_{n+1}.$$  

(c) Look up (or recall) the explicit expressions for the normalized harmonic oscillator energy eigenstates $\psi_2(x)$ and $\psi_3(x)$, then check the results of part (b) by applying the raising operator, expressed in terms of $d/dx$, to $\psi_2(x)$, and the lowering operator, similarly expressed, to $\psi_3(x)$.

3. For the simple harmonic oscillator, using the energy eigenfunctions as a basis, write each of the following operators explicitly in matrix form, showing enough rows and columns of each matrix to make the patterns clear: $H, a^-, a^+, x,$ and $p$. (I suggest doing them in the order listed. Notice that you can express $x$ and $p$ as linear combinations of $a^-$ and $a^+$.)

4. Consider a quantum system with two observable quantities $F$ and $G$, represented by the following operator matrices:

$$F = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (1)$$

(a) Show that $[F, G] = 0$, and conclude that these two observables must be compatible.

(b) Find the common eigenbasis of these two matrices.
(c) Suppose that this system is initially in the state \( \psi = (\sqrt{1/2}, 0, \sqrt{1/2}) \). You then measure \( F \), and obtain 2 as a result. What is the state vector after your measurement? (Hint: See the second paragraph under Principle 4 in Lesson 12.)

(d) Continuing the experiment described in part (c), suppose that you now measure \( G \). What values might you obtain, and what will be the system’s state vector after your measurement in each case?

(e) Still continuing this experiment, suppose that you now measure \( F \) again. Will you obtain the same result (2) as before? Will this measurement cause any further change to the state vector? Explain.

5. Suppose that an intelligent nonphysicist friend asks you to explain the uncertainty principle. Type up an explanation in your own words that is as precise as you can make it, while still being understandable to your friend and no longer than 300 words. Stick to the “original” version of the uncertainty principle, for position and momentum in one dimension.

6. For a three-dimensional wavefunction \( \psi(x, y, z) \) or \( \psi(r, \theta, \phi) \), the normalization condition is that the integral of \( |\psi|^2 \) over all of three-dimensional space must equal 1. Consider, then, the three-dimensional wavefunction \( \psi(r) = Ae^{-r/a} \), where \( r \) is the distance from the origin and \( A \) and \( a \) are constants. Supposing that the value of \( a \) is given, what is the normalization constant \( A \)? (You might as well assume that \( A \) is real and positive.)