

### 3. Time Evolution

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So far I’ve been discussing wavefunctions only as functions of space, at a single instant in time. What happens to a wavefunction as time passes?

The Einstein relation,  $E = hf$ , gives us a clue: If the wavefunction represents a state in which the particle has energy  $E$ , then it will oscillate in time at frequency  $f = E/h$  (or angular frequency  $\omega = E/\hbar$ ).

Most wavefunctions do *not* represent particles with well-defined energy, so this clue doesn’t tell us directly how they change over time. But for a *free* particle, with no potential energy (and therefore acted upon by no forces), the energy is entirely kinetic. In this case, the momentum eigenfunctions  $Ae^{ikx}$ , described in the previous lesson, must also be energy eigenfunctions, because knowing the momentum means you can calculate the energy.

And how exactly do we take the wavefunction  $Ae^{ikx}$  and make it oscillate in time at a given frequency? The answer is simple and elegant: we multiply it by  $e^{-i\omega t}$ , to obtain

$$\psi(x, t) = Ae^{ikx}e^{-i\omega t} = Ae^{i(kx - \omega t)}. \quad (1)$$

This is the general formula for the full time-dependent wavefunction of a free particle with well-defined momentum, and it takes the familiar form of a *traveling wave*: a function only of the combination  $kx - \omega t$ . The individual “ripples” in the wave move at the so-called phase velocity,

$$v_{\text{phase}} = \frac{\omega}{k} = \frac{E/\hbar}{p_x/\hbar} = \frac{E}{p_x}. \quad (2)$$

The time-dependent factor  $e^{-i\omega t}$  will come up repeatedly in this course, so let me give it a name: the *wiggle factor*. The wiggle factor must be a complex exponential (and not merely a sine or a cosine) in order to get the simple traveling-wave form in equation 1 (that is, a function only of the combination  $kx - \omega t$ ). The minus sign in the exponent of the wiggle factor is required in order for a positive- $k$  traveling wave to travel to the *right*. (However, the plus sign in  $e^{+ikx}$  was a mere convention, so we *could* have instead chosen a minus sign there and a plus sign in the wiggle factor. Nobody actually does it that way, so please stick to the standard sign convention.)

Equations 1 and 2 work for both nonrelativistic and relativistic particles—the only difference being the formula that relates the momentum  $p_x$  to the (kinetic) energy  $E$ . The simplest case is actually for an ultrarelativistic (massless) particle such as the photon, for which  $E = |p_x|c$  and therefore  $v_{\text{phase}} = \pm c$ , as you probably would have guessed. For a nonrelativistic particle, on the other hand, the result may surprise you. In this case, if the particle’s mass is  $m$ , then  $E = p_x^2/2m$  and we obtain

$$v_{\text{phase}} = \frac{p_x^2/2m}{p_x} = \frac{p_x}{2m} = \frac{v_x}{2}, \quad (3)$$

where  $v_x$  is the particle's velocity. The phase velocity of the wavefunction is only *half* the velocity of the particle! This may seem like nonsense, but it's not, because you don't measure a particle's velocity by measuring the phase velocity of the ripples within its wavefunction.

Now would be a good time to explore the Sinusoidal Wave animation that is linked from our course web page. Notice how the wavelength, frequency, and phase velocity all change as you adjust the momentum slider. Also, as a review of the previous lesson, notice the difference in the phase relationships between wavefunctions with positive and negative momentum. If you want some help visualizing how the phase velocity can be only half the particle velocity, take a look next at the Barrier Scattering simulation, setting the barrier width to zero so there's no potential energy (until the wavefunction hits the edge of the region).

We have now implemented both the Einstein and de Broglie relations, more precisely, in terms of the sinusoidal behavior of wavefunctions:

- When  $p_x$  is well defined,  $\psi \propto e^{ikx}$  with  $k = p/\hbar$  (de Broglie).
- When  $E$  is well defined,  $\psi \propto e^{-i\omega t}$  with  $\omega = E/\hbar$  (Einstein).

Although I've motivated the second statement by applying it to a free particle, it turns out to be completely general, even for systems with potential as well as kinetic energy.

For a single particle in one dimension, we will write the total energy as

$$E = K + V(x), \quad (4)$$

where the kinetic energy  $K$  is a function of momentum and the potential energy  $V(x)$  can be practically any function of position. When  $V = 0$ , the energy eigenfunctions are the same as the momentum eigenfunctions (that is,  $e^{ikx}$ ), as described above. But those functions describe particles whose positions are not well defined at all; conversely, the spiky position eigenfunctions describe particles whose momenta are not well defined. You might therefore think that when  $V \neq 0$  there would be *no* functions for which  $E = K + V$  is well defined, but in fact there always is a collection of such energy eigenfunctions—they're just different from both the position eigenfunctions and the momentum eigenfunctions. Finding the energy eigenfunctions for a given  $V(x)$  is not always easy, and in any case it must be done separately for each new function  $V(x)$ . We'll devote a lot of effort toward finding various energy eigenfunctions in this course, because they're so important.

For now, though, my main point about the energy eigenfunctions is that they always evolve in time according to the wiggle factor  $e^{-i\omega t}$ . So, for example, if the wavefunction at  $t = 0$  is  $\psi(x, 0)$  and if  $\psi(x, 0)$  is an energy eigenfunction with energy  $E$ , then

$$\psi(x, t) = \psi(x, 0) e^{-iEt/\hbar} \quad (\text{for an energy eigenfunction}). \quad (5)$$

This is just the general version of equation 1, which was specific to the case of a free particle.

What if  $\psi(x, 0)$  is not an energy eigenfunction? The next simplest case is when it is a mixture of just two energy eigenfunctions:

$$\psi(x, 0) = c_1\psi_1(x) + c_2\psi_2(x), \quad (6)$$

where  $\psi_1$  and  $\psi_2$  are energy eigenfunctions and  $c_1$  and  $c_2$  are constants. Then, as you might guess, the principle of superposition applies: Each contribution to the total wavefunction simply oscillates at its own frequency, so

$$\psi(x, t) = c_1\psi_1(x) e^{-iE_1t/\hbar} + c_2\psi_2(x) e^{-iE_2t/\hbar}, \quad (7)$$

where  $E_1$  and  $E_2$  are the energies corresponding to the eigenfunctions  $\psi_1$  and  $\psi_2$ .

Equation 7 generalizes in the natural way to mixtures of three or more energy eigenfunctions. Moreover, we'll see later that *any* wavefunction can be expressed as a linear combination of some number of energy eigenfunctions—although that number is sometimes infinite, so we might need to deal with an infinite sum. In the language of linear algebra, we say that the collection of energy eigenfunctions (for any potential energy  $V(x)$ ) always forms a complete basis, from which all other wavefunctions can be built.

We're now ready to turn to our first example of finding the energy eigenfunctions for a “trapped” particle, subject to a nonzero  $V(x)$ .