

## 14. Multiple Particles

Copyright ©2015–2016, Daniel V. Schroeder

I'll get right to the point:

**A system of two particles has only one wavefunction.**

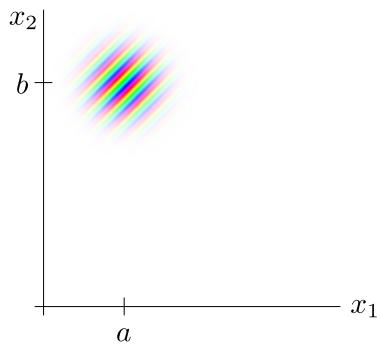
Read that sentence aloud. Repeatedly. It takes some getting used to.

And it gets worse: A system of three particles, or four, or  $10^{23}$ , also has only one wavefunction.

But let's start with just two particles, and say they're moving in just one dimension (to keep things as simple as possible). Then if  $x_1$  represents the position of the first particle and  $x_2$  represents the position of the second particle, the wavefunction is a function of two variables:  $\psi(x_1, x_2)$ . In fact, the allowed wavefunctions of this system are mathematically the same as those of a *single* particle in *two* dimensions: Just change  $(x, y)$  to  $(x_1, x_2)$ .

How do we interpret  $\psi(x_1, x_2)$ ? Well, if you integrate its square modulus over some range of  $x_1$  values *and* some range of  $x_2$  values, you should get the probability of finding the first particle in the first range *and* the second particle in the second range (if you measure both of their positions). If you just want the probability of finding one particle in a certain region, you have to integrate over *all* possible values of the other particle's coordinate. Again, just think about a single particle in two dimensions and change  $(x, y)$  to  $(x_1, x_2)$ .

We can visualize  $\psi(x_1, x_2)$  in the same way as  $\psi(x, y)$ , using a two-dimensional density plot. You just have to keep reminding yourself that the two directions on the plot correspond to the locations of the two particles along a single axis, rather than two different directions of space. Here, for example, is a plot of the wavefunction for a system in which particle 1 is localized around  $x = a$  and moving in the  $+x$  direction, while particle 2 is localized around  $x = b$  and moving in the  $-x$  direction:



The formula for this wavefunction would be something like

$$\psi(x_1, x_2) = e^{-(x_1-a)^2} e^{ikx_1} e^{-(x_2-b)^2} e^{-ikx_2}, \quad (1)$$

and as you can see, this function is separable: it factors into a function of  $x_1$  times a function of  $x_2$ .

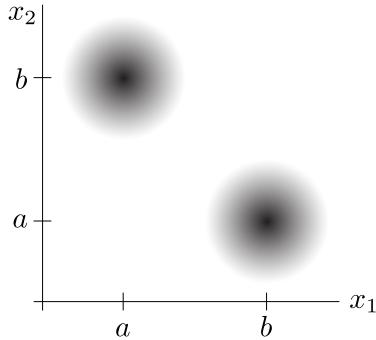
In general, a separable wavefunction is one for which

$$\psi(x_1, x_2) = \psi_a(x_1)\psi_b(x_2) \quad (\text{usually not possible}), \quad (2)$$

for some functions  $\psi_a$  and  $\psi_b$ . Whenever a two-particle system's wavefunction is separable, you can get away with saying that each particle has its own wavefunction. But just as we saw in the previous lesson, separability is the exception rather than the rule: the vast majority of all possible two-dimensional wavefunctions are not separable. We can easily build a non-separable wavefunction by adding two different separable functions, for example,

$$e^{-(x_1-a)^2}e^{-(x_2-b)^2} + e^{-(x_1-b)^2}e^{-(x_2-a)^2}, \quad (3)$$

which would look like this:



On one hand, this wavefunction might seem no more strange than a single-particle wavefunction that has two separated peaks. But please notice the following curiosity: The positions of both particles are indeterminate, so that if you measured either one of them, you would have a 50/50 chance of finding it either near  $a$  or near  $b$ . But if you do measure just one of them, the result will also tell you something about the other; for example, if you measure particle 1's position and find it near  $b$ , then you immediately know that particle 2 is near  $a$ .

Whenever a two-particle wavefunction is not separable, we say that the two particles are *entangled*. As the example above demonstrates, this means that you might be able to learn something about both particles by performing a measurement on just one of them.

Wavefunctions typically become entangled when the particles interact with each other. For example, although the wavefunction in equation 1 is not entangled, it does describe two particles that are moving toward each other. If these two particles interact, through a potential energy function that depends on their separation distance, then in all likelihood they will become entangled as time passes.

Ironically, even though the vast majority of multi-particle wavefunctions are *not* separable, we'll spend quite a bit of time working with wavefunctions that *are* separable. This is because they are mathematically simpler, and also because the separable wavefunctions are *complete*: any other wavefunction can always be expressed as a linear superposition of separable wavefunctions (as in equation 3). Just don't let these separable wavefunctions give you the false impression that the quantum world is always so simple. In quantum mechanics you need to keep reminding yourself that your basis vectors are merely a basis, from which you can build arbitrary linear combinations to represent the unlimited variety of the world.

The Hamiltonian of a two-particle system is, as usual, the operator that corresponds to the system's total energy. So it consists of a kinetic energy term for each particle, plus an arbitrary potential energy term that can depend on both coordinates:

$$H = -\frac{\hbar^2}{2m_1} \frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m_2} \frac{\partial^2}{\partial x_2^2} + V(x_1, x_2), \quad (4)$$

where  $m_1$  and  $m_2$  are the masses of the two particles. Depending on the system, the potential energy function can include "external" contributions that depend on the separate coordinates  $x_1$  and  $x_2$ , as well as an "internal" contribution that depends on the separation distance  $x_2 - x_1$ .

All of these ideas and formulas generalize in the natural way to systems of two particles in more than one dimension, and to systems of more than two particles. So, for instance, a system of six particles in three dimensions, such as the electrons in a carbon atom, would have a wavefunction that depends on 18 different variables:  $x_1, y_1, z_1, x_2, \dots, z_6$ . Much of the art of quantum mechanics consists in figuring out how many of the variables really need to be treated quantum mechanically, and then modeling the system in a way that's simple enough to make calculations feasible and to provide genuine insight.

## Identical particles

Quantum mechanics allows two (or more) particles to be *identical* in a sense that isn't really possible in classical mechanics. Consider what happens when two particles have all the same intrinsic properties (mass, electric charge, spin orientation, etc.), and in addition, their two-particle probability density is unchanged under the operation of interchanging its two arguments:

$$|\psi(x_2, x_1)|^2 = |\psi(x_1, x_2)|^2. \quad (5)$$

If this relation holds, then the two particles can't even be distinguished by their locations or velocities (or by the probabilities of their being found at various locations or with various velocities), so it becomes meaningless to even ask which one is which. This situation is fundamentally different from classical mechanics, in which you can always distinguish particles by their trajectories (if nothing else).

Moreover, it's not hard to prove that if the particles are identical and equation 5 holds at one particular time, then the time-dependent Schrödinger equation will ensure that it holds at all other times.

It seems that nature has taken advantage of this opportunity, and filled the universe with particles that are truly identical to each other in the sense of equation 5. All electrons are identical to each other in this sense, as are all protons, and all alpha particles, and all photons, and even all carbon-12 atoms (in their ground state). There are actually two different ways that equation 5 can be realized:

$$\psi(x_2, x_1) = +\psi(x_1, x_2) \quad \text{or} \quad \psi(x_2, x_1) = -\psi(x_1, x_2). \quad (6)$$

(Equation 5 would also be satisfied if  $\psi$  changed by an arbitrary phase  $e^{i\phi}$  when  $x_1$  and  $x_2$  are interchanged, but consistency requires that repeating the interchange operation restore the wavefunction to its original value, so  $\phi$  can be only 0 or  $\pi$ .) Again, the TDSE ensures that the choice of + or - in equations 6 is preserved as time passes, so every species of particles falls into one of two categories, according to this difference. Particles for which  $\psi(x_2, x_1) = +\psi(x_1, x_2)$  are called *bosons* (after Satyendra Nath Bose), and include photons, pions, helium-4 nuclei, and many other species of nuclei and atoms. Particles for which  $\psi(x_2, x_1) = -\psi(x_1, x_2)$  are called *fermions* (after Enrico Fermi), and include electrons, protons, neutrons, helium-3 nuclei, and many other species of nuclei and atoms. You can tell which particles are bosons and which are fermions by looking at their intrinsic spins, which I'll discuss further in the next lesson.

Equations 6 are the most general ways of expressing how identical-particle wavefunctions behave under interchange, but as usual, it's more vivid to consider two-particle wavefunctions that are built out of single-particle wavefunctions. Suppose, for example, that we have a two-particle system in which one particle is in state  $\psi_a(x)$  while the other particle is in state  $\psi_b(x)$ . Then, if the two particles are bosons, the wavefunction of the system would be (up to an overall normalization constant)

$$\psi(x_1, x_2) = \psi_a(x_1)\psi_b(x_2) + \psi_a(x_2)\psi_b(x_1), \quad (7)$$

while if the two particles are fermions, the wavefunction of the system would be (again up to an overall normalization constant)

$$\psi(x_1, x_2) = \psi_a(x_1)\psi_b(x_2) - \psi_a(x_2)\psi_b(x_1). \quad (8)$$

In both cases, you can immediately see that conditions 6 are satisfied. Notice, by the way, that if  $\psi_a$  and  $\psi_b$  are actually the *same* state, then equation 7 still makes perfect sense (and reduces to a simple product), but equation 8 equals zero—which isn't allowed. It's therefore impossible to put two identical fermions into the same single-particle state; this fact is called the *Pauli exclusion principle*.