13. Multiple Spatial Dimensions

We're now ready to generalize quantum mechanics beyond the system of a single, structureless particle in one dimension. Specifically, we will consider:

- particles moving in multi-dimensional space;
- systems of multiple particles; and
- particles with internal degrees of freedom (such as spin).

In this lesson we'll consider a single particle moving in two spatial dimensions, with occasional remarks about the further generalization to three dimensions.

In many ways, the generalization from one to two dimensions is completely straightforward. The wavefunction ψ is now a complex-valued, square-integrable function of *two* variables, x and y, normalized so that

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, |\psi(x,y)|^2 = 1. \tag{1}$$

To compute the probability of finding the particle within some two-dimensional region, we would integrate $|\psi|^2$ only over that region.

If a particle is localized, then its wavefunction is nonzero only over a small range of x and y values. In the idealized limit, such a function becomes a position eigenfunction, which is nonzero everywhere except at a single point, and can be expressed as a product of delta functions:

$$\delta(x-x_0)\delta(y-y_0) = \delta^{(2)}(\vec{r}-\vec{r}_0), \tag{2}$$

where $\vec{r} = (x,y)$ and the superscript (2) denotes a two-dimensional delta function. As in one dimension, however, a position eigenfunction is not normalizable.

An idealized momentum eigenfunction, on the other hand, would have the form

$$e^{ip_x x/\hbar} e^{ip_y y/\hbar} = e^{i\vec{p}\cdot\vec{r}/\hbar}, \tag{3}$$

where $\vec{p} = (p_x, p_y)$. This function varies only as $\vec{r}$ changes in the $\vec{p}$ direction. Each wave "crest" or "trough" consists of a straight line, perpendicular to $\vec{p}$, so we call this a *linear wave*. In three dimensions the crests and troughs would be planes, so we would call $e^{i\vec{p}\cdot\vec{r}/\hbar}$ a *plane wave*.

To visualize a wavefunction in two dimensions, it's usually easiest to plot the two dimensions horizontally and vertically, and then represent the wavefunction value using colors or gray levels. For example, here is a picture of a two-dimensional Gaussian wavepacket with a rightward (average) momentum, in which the saturation represents the probability density and the hues represent phases as usual:

The formula for this Gaussian wavepacket would be

$$\psi(x,y) = e^{-(x^2+y^2)/a^2} e^{i\vec{p}\cdot\vec{r}/\hbar}, \tag{4}$$

where $\vec{p}$ points to the right and I've neglected an overall normalization constant.

(Visualizing *three*-dimensional wavefunctions is quite a bit more difficult, but there's some cool software at `falstad.com` that can help.)

In more abstract language, the state "vectors" of the system now live in a different (larger) vector space than for a one-dimensional system. The normalization integral (equation 1) is just a special case of the inner product of two wavefunctions, whose general definition in two dimensions is

$$\langle \psi_1, \psi_2 \rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \psi_1^*(x,y)\psi_2(x,y). \tag{5}$$

In this vector space there are two independent position operators, x and y, which we can think of as components of a two-dimensional vector position operator $\vec{r}$. Similarly, there are two momentum operators,

$$-i\hbar\frac{\partial}{\partial x} \quad \text{and} \quad -i\hbar\frac{\partial}{\partial y}, \tag{6}$$

which we can think of as components of a two-dimensional vector momentum operator, $-i\hbar\nabla$. The sinusoidal linear waves in equation 3 are eigenfunctions of these operators, with the expected eigenvalues p_x and p_y.

To find the Hamiltonian operator for this two-dimensional system, we simply start with the classical expression for the system's total energy and substitute operators for the momentum components:

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + V(\vec{r}) \longrightarrow -\frac{\hbar^2}{2m}\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) + V(\vec{r}). \tag{7}$$

The differential operator inside the parentheses is often abbreviated ∇^2. With this Hamiltonian you can now write down either the TISE ($H\psi = E\psi$) or the TDSE ($H\psi = i\hbar\,\partial\psi/\partial t$) for this two-dimensional system.

All of the wavefunction formulas in equations 2, 3, and 4 have the special property that they can be *factored* into a function of x times a function of y:

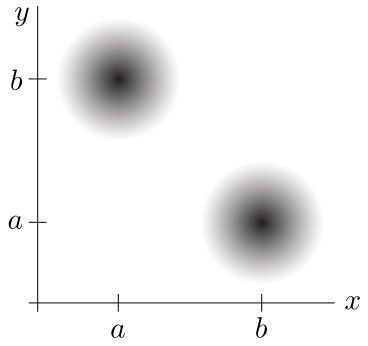
$$\psi(x,y) = \psi_x(x)\psi_y(y) \qquad \text{(usually not possible)}. \tag{8}$$

Such functions are said to be *separable* (in rectangular coordinates). However, this property is the exception rather than the rule. It's easy to write down wavefunction formulas, such as $e^{-x^2y^2/a^4}$ and $e^{ip\sqrt{x^2+y^2}/\hbar}$, that are not separable.

One especially easy way to construct non-separable wavefunctions is to combine two or more separable wavefunctions into a superposition. For example, consider the superposition of Gaussians

$$e^{-(x-a)^2}e^{-(y-b)^2} + e^{-(x-b)^2}e^{-(y-a)^2}, \tag{9}$$

where x and y are understood to be measured as multiples of some suitable unit. Each term in this expression is separable, but the entire expression (which is an equally legitimate wavefunction) is not. The function looks something like this:



On one hand, this wavefunction is no more strange than a one-dimensional wavefunction that has two separated peaks; loosely speaking, we could say that the particle is half near one location and half near the other. But please notice the following further curiosity: Both the x and y coordinates of this particle are indeterminate, so that if you measured either of them, you would have a 50/50 chance of finding it either near a or near b. But if you do measure just one of them, the result will also tell you something about the other; for example, if you measure x and find it near b, then you immediately know that y is near a.

When a wavefunction depends on two variables but is not separable, we often say that the two variables are *entangled*. As the example above demonstrates, this means that you might be able to learn something about both variables by measuring just one of them.

I should point out that sometimes a non-separable wavefunction becomes separable if you switch to a different coordinate system. For example, the function $e^{ip\sqrt{x^2+y^2}/\hbar}$ is separable in polar coordinates, while the wavefunction of equation 9

can be factored if you rotate the coordinate axes by 45 degrees. Often the relevant coordinate system is dictated by some other outside condition, so a change of coordinates may or may not be appropriate. And if you make these wavefunctions just a little more complicated, e.g., by adding a third Gaussian peak to equation 9, then they can't be factored in any coordinate system.

Ironically, even though the vast majority of wavefunctions are *not* separable, we'll spend most of our time working with wavefunctions that *are* separable. This is because they are mathematically simpler, and also because the separable wavefunctions are *complete*: any other wavefunction can always be expressed as a linear superposition of separable wavefunctions (as in equation 9). Just don't let these separable wavefunctions give you the false impression that the quantum world is always so simple. In quantum mechanics you need to keep reminding yourself that your basis vectors are merely a basis, from which you can build arbitrary linear combinations to represent the unlimited variety of the world.