

15. Particles with Internal Structure

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Our third and final generalization of quantum mechanics is to particles that have internal degrees of freedom. The state ψ of such a particle depends not only on its location in space, but also on one or more internal variables.

The most familiar type of internal degree of freedom is *spin*, a particle's intrinsic angular momentum. But there are many other examples. Atoms can be internally excited above their ground states, and molecules can be in various rotational and vibrational states. Quarks and gluons each have several “color” states, and the quarks also have various “flavor” states (up, down, strange, charm, top, bottom), as do the leptons (electron, muon, tau, and three neutrino varieties). Even the familiar proton and neutron can often be considered as two different states of a single “nucleon” system.

One common feature of all these types of internal states is that they are discrete, rather than continuous. This means that the “wavefunction” that describes a particle's internal state is not a function of a continuous variable like x , but rather a function of a discrete variable that takes on only certain values. For atomic and molecular excited states the list of values is still nominally infinite, but in many cases, including spin, the list is finite, and in the simplest cases there are just two possible values. For example, the spin state of a “spin-1/2” particle like the electron (or proton or neutron or any of the quarks or leptons) can be described as a function of a parameter called m_s whose value must be either $+1/2$ or $-1/2$.

Still taking the spin state of an electron as an example, we *could* denote this state by a symbol like $\psi(m_s)$, in analogy to the spatial wavefunction $\psi(x)$. But because m_s can have only two possible values, we normally don't put it in parentheses like a function argument; instead we write ψ as a two-component column vector (sometimes called a *spinor*) whose (complex) components are $\psi(1/2)$ and $\psi(-1/2)$, respectively:

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}. \quad (1)$$

In this expression I've also introduced the more common notations $\psi_{\pm}$ for $\psi(\pm 1/2)$. The interpretations of these components are the natural counterparts to the interpretation of $\psi(x)$:

$$\left(\begin{array}{c} \text{Probability of finding} \\ \text{particle with } m_s = \pm 1/2 \end{array} \right) = |\psi_{\pm}|^2. \quad (2)$$

No integration is necessary, because the values of m_s are discrete. Physically, by the way, m_s equals the z component of the electron's angular momentum, in units of $\hbar$. Often we refer to the electron's two spinor components simply as “spin up” and “spin down.”

Of course the sum of the probabilities for all possible values of m_s must equal 1, so there is a normalization condition:

$$|\psi_+|^2 + |\psi_-|^2 = 1 \quad \text{or} \quad \langle \psi, \psi \rangle = 1, \quad (3)$$

where the inner product is defined as the familiar vector dot product, with a complex conjugation on the first of the two vectors. In matrix notation,

$$\langle \psi_1, \psi_2 \rangle = (\psi_{1+}^*, \psi_{1-}^*) \begin{pmatrix} \psi_{2+} \\ \psi_{2-} \end{pmatrix}. \quad (4)$$

But I've been ignoring the motion of the electron through space. Its full wavefunction is a function of x , y , z , and m_s . But again, instead of writing $\psi(x, y, z, m_s)$, we express the m_s dependence using spinor notation:

$$\psi = \begin{pmatrix} \psi_+(x, y, z) \\ \psi_-(x, y, z) \end{pmatrix}. \quad (5)$$

In general, the two components of the spinor can be completely different functions of x , y , and z . In the simplest cases, though, the wavefunction factors (or separates) into a single function of space times a space-independent spinor. When it *doesn't* factor, we say that the particle's spatial motion and its spin are entangled. (Yes, in principle you can put an electron into a state with its spin-up component in Albuquerque and its spin-down component in Denver.)

For the more general case in which the internal state of a particle depends on a parameter that has n possible values, we describe that internal state using an n -dimensional complex-valued column vector. Some particles have three or more independent spin states (m_s values); gluons come in eight colors; and the number of vibrational states of a diatomic molecule is nominally infinite (though in practice the molecule will break apart if the high- n states are active). In all cases, the internal states live in an n -dimensional vector space, and if we also wish to describe the particle's spatial motion, then the overall states live in the vector space of n -tuples of functions of space.

Sometimes a single particle has two types of internal degrees of freedom—such as a quark with two independent spin states (+ and −) and three independent colors (r , g , and b). To describe both the spin and color of a quark we could use a *six*-dimensional column vector, associating the components with $r+$, $r-$, $g+$, $g-$, $b+$, and $b-$. More often, though, we work with “factored” basis states that keep spin and color separate, and simply juxtapose the respective column vectors in what is called a *direct product*:

$$\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \begin{pmatrix} \psi_r \\ \psi_g \\ \psi_b \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \otimes \begin{pmatrix} \psi_r \\ \psi_g \\ \psi_b \end{pmatrix}. \quad (6)$$

This notation is convenient because inner products and other matrix multiplications can also be kept within the separate spin and color “spaces”; you just manipulate the spin and color expressions separately, ignoring each while working with the other. The down-side is that the only way to write down a state with spin and color entangled is as a sum of terms that are separately factored.

And the same principles apply when you want to keep track of the internal states of two (or more) different particles. You can write down non-entangled states by juxtaposing column vectors for the separate particles, but linear combinations (that is, entanglements) of these states are also possible, and you need to remember that you’re really working in the vector space of all direct products *and* their linear combinations.

We’ll see some concrete examples of these ideas later in the course, when you’ll have more time to practice with them. For now, my main goal is to warn you about what’s coming up, and to give you a general sense of how we can combine any two separate quantum systems into a larger composite system by working in an appropriately larger vector space.