18. Compatible and Incompatible Observables

Suppose you measure the position of a quantum particle, and then you measure its momentum, and then you measure its position again. Chances are that your second position measurement will be quite different from your first, because the intervening momentum measurement put the particle into a momentum eigenstate, spread out widely in space, effectively erasing all memory of the outcome of your first position measurement. We therefore say that position and momentum are *incompatible observables*.

Similarly, in the ABC Laboratory simulation you can easily check that if you measure A, then measure B, and then measure A again, there's a good chance that the second A measurement will give a result that's different from the first. Thus, A and B are also incompatible observables.

An example of *compatible* observables would be momentum and kinetic energy: measuring one of these quantities will have no effect on subsequent measurements of the other. In general, two observables are compatible if you can measure one, then measure the other, then measure the first again, and be *guaranteed* of getting the same result in the final measurement that you got in the first one. We'll later see that the magnitude of a particle's angular momentum is compatible with any one component of its angular momentum, and that both of these are compatible with energy whenever a particle is subject to a spherically symmetric potential energy function.

Mathematically, there are two ways to characterize whether observables are compatible (or incompatible). First, we can ask whether the operators for the two observables possess a common eigenbasis (that is, a common set of eigenvectors that form a complete basis). Second, we can ask whether the operators themselves commute with each other. In summary, the following three statements are all equivalent:

1. A and B are compatible observables.
2. The A and B operators possess a common eigenbasis.
3. The A and B operators commute.

I won't try to write out a mathematician's proof of these equivalencies, but it's not hard to understand the basic ideas. I'll first explain why (1) and (2) are equivalent, then why (2) and (3) are equivalent.

Suppose, first, that A and B are compatible observables, and imagine measuring A first, then B, and then A again. The first measurement puts the system into one of the A eigenstates, and the second measurement (of B) must leave the system in an A eigenstate with the same eigenvalue, so that the final measurement is sure to yield the same outcome as the first. But this means that this A eigenstate is also a B eigenstate. In the simplest cases this final eigenstate will be the same one that the initial measurement of A put the system into. There's a complication, though, if the value of A obtained is degenerate, so that more than one linearly independent eigenvector of A has this eigenvalue. Then if B lacks this degeneracy, the B measurement will usually change the state to a *different* vector in the subspace of A-degenerate eigenstates. In any case, however, we can find *some* set of basis vectors that are eigenvectors of *both* A and B.

Conversely, if A and B have the same eigenvectors then measuring A puts the system into an eigenstate of both, so a subsequent measurement of B doesn't change this state and therefore doesn't affect a subsequent measurement of A. Degeneracy again complicates this picture somewhat, but doesn't alter the conclusion that the two observables must be compatible.

Once we know that A and B have a common eigenbasis, we can see that they must commute by applying them in succession to an arbitrary vector ψ that is expanded in terms of this common eigenbasis. If the basis vectors are $\{\alpha_n\}$ and the expansion coefficients are $\{c_n\}$, then

$$AB\psi = AB\sum_n c_n\alpha_n = \sum_n c_n AB\alpha_n = \sum_n c_n a_n b_n \alpha_n, \tag{1}$$

where a_n and b_n are the associated eigenvalues of A and B, respectively. But applying A and B to ψ in the other order (that is, $BA\psi$) would give exactly the same expression, and therefore A and B commute.

Conversely, if we assume that A and B commute, then we can prove that if α is an eigenvector of A with eigenvalue a, then $B\alpha$ is also an eigenvector of A with the same eigenvalue:

$$A(B\alpha) = B(A\alpha) = B(a\alpha) = a(B\alpha). \tag{2}$$

If the eigenvalue a is nondegenerate, then this means that $B\alpha$ must be proportional to α itself, so α is also an eigenvector of B. In the degenerate case the vector $B\alpha$ could lie along some different direction in the subspace of degenerate eigenvectors of A, but there must always be a set of basis vectors in this subspace that are also eigenvectors of B.[1]

The uncertainty principle

When two observables are incompatible, we might still wonder *how much* a measurement of one of them interferes with measuring the other. After all, in the classical limit there is no such thing as incompatibility: measurements don't affect the state of the system at all.

[1]For a proof of this statement, see F. Mandl, *Quantum Mechanics* (Wiley, Chichester, 1992), Section 3.1.

To quantify the degree of incompatibility, however, it's useful to rephrase the question. Instead of considering a succession of measurements (A then B then A again), let's ask whether we can find at least *some* states for which A and B are both *approximately* well defined. More precisely, imagine preparing a large number of identical systems in the same state, then measuring A for half of these systems and B for the other half. Each set of measurements will have some average value, $\langle A \rangle$ or $\langle B \rangle$, and will also have some amount of spread about the average, which we can characterize by the standard deviation, σ_A or σ_B. We say that a quantity is approximately well defined if its standard deviation is small.

In general, there is no limit on how small we can make σ_A or σ_B: either could be zero, if the state in question is an eigenstate of that observable. And if A and B are compatible, then we can find simultaneous eigenstates for which *both* σ_A and σ_B are zero. But when A and B are incompatible, there is generally a limit on how small we can simultaneously make σ_A and σ_B. Specifically, one can prove that the product $\sigma_A \sigma_B$ must obey the inequality

$$\sigma_A \sigma_B \geq \left| \frac{1}{2i} \langle [A, B] \rangle \right|. \tag{3}$$

This is the most general version of the famous *uncertainty principle.* It says that there is *no* state for which the product of standard deviations is larger than the right-hand side. Notice that the right-hand-side involves the average value of the commutator of the operators A and B; for compatible observables, of course, this commutator would be zero so there would be no constraint on $\sigma_A \sigma_B$.

The proof of the generalized uncertainty principle is rather technical, so I'll simply refer you to Griffiths (Section 3.5).

For the special case of position and momentum (in one dimension), the commutator is simply $i\hbar$ so the generalized uncertainty principle reduces to

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}. \tag{4}$$

This is the original uncertainty principle that we all know and love. For macroscopic objects, of course, the value of $\hbar$ is so small that the uncertainty principle puts no practical constraint on precision with which position and momentum can be simultaneously well defined.