Phonograph  
(a) \( \alpha = \frac{\Delta \omega}{\Delta t} = \frac{\omega - \omega_1}{0.5 \text{ min}} = -33 \frac{\text{rev}}{\text{min}^2} \)  

\[ = -66 \frac{\text{rev}}{\text{min}^2} \]

If we take the original direction of rotation to be +,  

In more usual units,  
\[ \alpha = (-66 \frac{\text{rev}}{\text{min}^2}) \left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right) \left(\frac{1 \text{ min}}{60 \text{ s}}\right)^2 = -116 \frac{\text{rad}}{\text{s}^2}. \]

(b) In analogy to  
\[ x = x(0) + v(0)t + \frac{1}{2}at^2 \]

\[ \theta = \theta(0) + \omega(0)t + \frac{1}{2}\alpha t^2 \]

for angle  \( \theta \)  

\[ \rightarrow \theta - \theta(0) = (33 \frac{1}{2} \text{ rev})(0.5 \text{ min}) - \frac{1}{2} \left(66 \frac{2}{3} \text{ rev}^{-1} \text{min}^{-2}\right)(0.5 \text{ min})^2 \]

\[ = 8 \frac{1}{2} \text{ revolutions}. \]

Dumbbell  

\[ I = \sum \frac{m_i r_i^2}{\text{ pieces}} = (1 \text{ kg})(0.2 \text{ m})^2 + (1 \text{ kg})(0.2 \text{ m})^2 = 0.08 \text{ kg} \cdot \text{m}^2. \]

(b)  
\[ K = \frac{1}{2} I \omega^2 = \frac{1}{2} \left(0.08 \text{ kg} \cdot \text{m}^2\right) \left(\frac{2\pi \text{ rad}}{\text{s}}\right)^2 = 14.2 \text{ J}. \]

(c) Speed  
\[ \frac{2\pi r}{T} = \frac{2\pi \cdot (2 \text{ m})}{\frac{2\pi}{360 \text{ s}}} = 3.77 \text{ m/s}. \]

(d)  
\[ K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} (1 \text{ kg})(3.77 \text{ m/s})^2 \cdot 2 = 14.2 \text{ J}. \]

(same!)
Moments of Inertia

The greatest rotational inertia is when the mass is farthest from the rotational axis — that would be the hoop.

The least rotational inertia is when the mass is in close to the axis: the sphere.

Bicycle Crank

(a) \( T = \frac{10}{11} \frac{1}{2} \sin \phi \)

\[ = (1.52 \text{ m})(11 \text{ N}) \sin 150^\circ \]

\[ = 8.44 \text{ N.m} \]

(b) At 90°, \( \vec{F} \) and \( \vec{r} \) are \( \perp \), so \( T = \frac{10}{11} \frac{1}{2} = 16.9 \text{ Nm} \).

(c) At 180°, \( \vec{F} \) and \( \vec{r} \) are parallel, so \( T = 0 \).

Torque Practice

Torque from \( \vec{F}_A \): \( (8 \text{ m})(10 \text{ N}) \sin 135^\circ = 56.6 \text{ N.m} \) (ccw)

Torque from \( \vec{F}_B \): \( (4 \text{ m})(16 \text{ N}) \sin 90^\circ = 64.0 \text{ N.m} \) (clockwise)

Torque from \( \vec{F}_C \): \( (3 \text{ m})(19 \text{ N}) \sin 160^\circ = 19.5 \text{ N.m} \) (ccw)

So the net torque in the ccw direction is

\[ \sum T = (56.6 \text{ N.m}) - (64.0 \text{ N.m}) + (19.5 \text{ N.m}) \]

\[ = 12.1 \text{ N.m} \]
Massive Door

\[ I = \sum m_i r_i^2 \]

If we pretend that all pieces are at \( r = 1.2 \text{ m} \) (half the width), we get

\[ I \approx M r^2 = (44,000 \text{ kg})(1.2 \text{ m})^2 = 63,400 \text{ kg} \cdot \text{m}^2 \]

which is somewhat low but not so far off.

A steady force produces a constant \( \alpha \), so

\[ \Delta \theta = \theta - \theta_0 = \frac{5 \alpha}{2} \cdot t + \frac{1}{2} \alpha t^2 \rightarrow \alpha = \frac{2 \Delta \theta}{t^2} \]

The required torque is \( I \alpha \), and the required force is \( \frac{F}{r} \)

(since \( \theta = 90^\circ \)). So:

\[ \begin{align*}
    |F| = \frac{F}{r} = \frac{I \alpha}{r} = \frac{I}{r} \cdot \frac{2 \Delta \theta}{t^2} = \frac{(8.7 \times 10^5 \text{ kg} \cdot \text{m}^2) \cdot 2 \left( \frac{\pi}{2} \right)}{(2 \pi)(30 \text{ s})^2} \\
    = 126 \text{ kN} \cdot \text{m} = 126 \text{ N}. \quad \text{Not so very much!}
\end{align*} \]

At \( r = 1.2 \text{ m} \), we’d need twice the force, 252 N.

Rolling Wheel

\[ \begin{align*}
    (a) & \quad x(t) = x(0) + \frac{1}{2} a_t t^2 \\
        & \quad v(t) = \frac{dx}{dt} \quad \left\{ \begin{array}{l}
            x(0) = v(0) \left( \frac{x(0)}{a_x} \right) + \frac{1}{2} a_v \left( \frac{x(0)}{a_x} \right)^2 \\
            v(t) = v_0 + a_v t \\
            a_x = -\frac{v(t)}{2 x(t)} = -\frac{(43 \text{ m/s})}{2 \cdot 225 \text{ m}} = -4.11 \text{ m/s}^2 \quad \text{(slows down)}
        \end{array} \right.
    \\
    \text{So} & \quad a_x = \frac{a_x}{t} = \frac{4.11 \text{ m/s}^2}{2.25 \text{ m}} = 16.4 \text{ s}^2.
\end{align*} \]

(b) With respect to the axis, the ground and the wheel rim have a linear acceleration of 4.11 m/s. (never mind the direction).
Conservation Laws Worksheet

1. Pictorial Representation

   \[ \text{INITIAL} \]
   \[ \text{FINAL} \]

   Assume \( r \ll R \), so \( y_f = 2R \)

   List known quantities:
   \[ R, g \]
   Ball is solid, uniform
   \[ I = \frac{2}{5} mr^2 \]

   List unknown quantities:
   \[ h, \text{ everything else} \]

2. Physical Representation

   Specify what the "system" is:
   The ball (and, technically, the earth + the track)

   Which conservation law(s) apply, and why?
   Energy is conserved.
   Gravity is a conservative force \( (U = mg(y)) \).
   Normal force does no work.
   No sliding \( \Rightarrow \) no kinetic friction.
   (Static friction causes ball to roll but does not convert energy to thermal form.)

3. Mathematical Representation

   Write the appropriate conservation law(s) in mathematical form and solve.

   \[ E_{\text{initial}} = E_{\text{final}} \]

   \[ mgh = mg(2R) + \frac{1}{2} m v_f^2 + \frac{1}{2} I \omega^2 \]

   \[ h = 2R + \frac{\left| \frac{\dot{v}_f^2}{2g} \right| + \frac{1}{2mg} \cdot \frac{2}{5} mr^2 \omega^2}{\omega} \]

   \[ = 2R + \frac{\left| \frac{\dot{v}_f^2}{2g} \right| + \frac{1}{5} v_f^2}{g} (\text{no sliding}) \Rightarrow \left| \frac{\dot{v}_f}{g} \right| = r \omega \]

   \[ = 2R + \frac{7}{10} \left| \frac{\dot{v}_f^2}{g} \right| . \]

   Thus, \( h_{\text{min}} = 2R + \frac{7}{10} \frac{Rg}{g} \)

   \[ = (2.7) R \]

   Draw force diagram for top of loop:

   \[ \sum F_x = ma_y \]

   \[ |F_x| + |F_y| = m \frac{\dot{v}_f^2}{R} . \]

   Minimum \( |F_x| \) is when \( |F_y| = 0 \).

   \[ \text{so } mg = m \frac{\dot{v}_f^2}{R} \Rightarrow \dot{v}_f^2 = Rg \]

4. Evaluation

   \[ \checkmark \text{ correct sign?} \]
   \[ \checkmark \text{ appropriate units?} \]
   \[ \checkmark \text{ reasonable magnitude?} \]
As an object moves along in a straight line at constant speed, its angular momentum doesn't change. Why? Because

\[ L = m|\vec{v}| \sin \phi \]

and \( |\vec{v}| \) is constant, and \( r \sin \phi \) is the perpendicular distance from the origin to the path, which is also constant.

Alternatively, an object just coasting along has no net torque on it, so its angular momentum can't possibly change.

So: For \( \theta_1 \), at Harvard + 36° R,

\[ L = m|\vec{v}| r \perp = m|\vec{v}| \cdot 0 = 0 \]

For \( \theta_2 \), at 25° R + Washington,

\[ L = m|\vec{v}| r \perp = (1200 \text{ kg}) (40 \text{ mi}) (\frac{1 \text{ mi}}{264 \text{ mph}}) (1.2 \text{ mi}) \times (\frac{1609 \text{ m}}{1 \text{ mi}}) \]

\[ = 4.1 \times 10^7 \text{ kg} \cdot \text{m}^2 \text{ s}^{-1} \] (Positive)

For \( \theta_3 \), at Lind Lecture Hall,

\[ L = -m|\vec{v}| r \perp = - (half as much) \]

\[ = -2.07 \times 10^7 \text{ kg} \cdot \text{m}^2 \text{ s}^{-1} \]
Conservation Laws Worksheet

Collapsing Sun

(Adapted from Van Heuvelen, Overview Case-Study Physics)

1. Pictorial Representation

   INITIAL
   \( I_i \)
   \( M = \text{mass} \)
   \( \omega_i \)
   \( R_i \)

   FINAL
   \( I_f \)
   \( \omega_f \)
   \( R_f \)
   \( = \text{radius} \)

List known quantities:

\[ T_i = 25 \text{ days} \]
\[ R_i = 100 R_f \]

List unknown quantities:

\[ T_f = ? \]

2. Physical Representation

Specify what the "system" is:

The sun.

Which conservation law(s) apply, and why?

No external torques

\( \Rightarrow \) angular momentum is conserved.

3. Mathematical Representation

Write the appropriate conservation law(s) in mathematical form and solve.

\[ L_f = L_i \]

\[ \Rightarrow I_f \omega_f = I_i \omega_i \]

\[ \Rightarrow I_f \cdot \frac{2\pi}{T_f} = I_i \cdot \frac{2\pi}{T_i} \]

\[ \Rightarrow T_f = \frac{I_f}{I_i} \cdot T_i . \]

Thus

\[ T_f = \left( \frac{R_f}{R_i} \right)^2 T_i \]

\[ = \left( \frac{1}{100} \right)^2 (25 \text{ days}) \]

\[ \approx 4 \text{ minutes} . \]

Let's say the sun is a uniform solid sphere both before and after. Then

\[ \frac{I_f}{I_i} = \frac{\frac{2}{3} M R_f^2}{\frac{2}{5} M R_i^2} = \left( \frac{R_f}{R_i} \right)^2 . \]

In fact the sun's density is far from uniform, so the \( \frac{2}{5} \) isn't correct. So this is just a rough estimate.

4. Evaluation

\( \checkmark \) correct sign?

\( \checkmark \) appropriate units?

\( \checkmark \) reasonable magnitude?
1. Pictorial Representation

- Symbols, including masses and velocities
- Coordinate axes not needed

INITIAL

FINAL

List known quantities:
- \( m = 44 \text{ kg} \)
- \( M = 180 \text{ kg} \)
- \( R = 1.2 \text{ m} \)
- \( r = \text{ radius of gyration} = 0.71 \text{ m} \)
- \( v_c = 3 \text{ m/s} \)

List unknown quantities:
- \( I \), \( \omega \)

2. Physical Representation

Specify what the "system" is: child plus merry-go-round.

Which conservation law(s) apply, and why?

Angular momentum: no external torques (reflecting friction)

3. Mathematical Representation

(a) First calculate

\[ I = Mr^2 = (180 \text{ kg})(0.71 \text{ m})^2 = 149 \text{ kg m}^2. \]

(b) Child's initial

\[ I = mR^2 \sin \phi = mRv_c \]

\[ = (44 \text{ kg})(1.2 \text{ m})(3.5) \]

\[ = 158.4 \text{ kg m}^2. \]

(c) \[ L_{final} = L_{initial} \]

\[ \rightarrow I_{child} + I_{m.g.m.} = I_{m.g.m.} \]

\[ \rightarrow mRv_c + I \omega = I_{initial} \]

\[ \rightarrow mR^2 \omega + I \omega = L_{initial} \]

\[ \rightarrow \omega = \frac{L_{initial}}{mR^2 + I} \]

\[ = \frac{158.4 \text{ kg m}^2}{(44 \text{ kg})(1.2 \text{ m})^2 + 149 \text{ kg m}^2} \]

\[ = 0.75 \text{ s} \]

\[ \approx 0.75 \text{ rad s}^{-1} \]

(only \( 12 \) revolutions per second)

4. Evaluation

- correct sign?
- appropriate units?
- reasonable magnitude?