

Constrained Motion Problems

Physics 2210 lecture notes for 1 February 2008

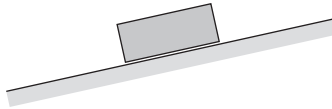
You've now spent about a week learning how forces cause objects to accelerate. You've learned the precise definitions of force and mass, and the precise statement of Newton's second law:

$$\vec{a} = \frac{\sum \vec{F}}{m}, \quad \text{or} \quad \sum \vec{F} = m\vec{a}.$$

That is, the acceleration vector (\vec{a}) of any object is determined by the vector sum of all the forces (\vec{F} 's) acting on that object, and is numerically equal to this sum of forces divided by the object's mass (m). You've also learned about most of the familiar *types* of forces, and you have formulas that allow you to calculate the gravitational force on any object and to relate the slipping or gripping friction force on an object to the "normal" (compression) force exerted by the same surface. You've learned that every force is exerted by a nearby object (the "agent" of the force). And finally, you've developed a powerful conceptual tool, the *force diagram*, for visualizing the forces acting on any object.

Now it's time to put all this knowledge to work solving a variety of problems. Although Newton's second law can be used to predict the motion of an object along any path through three-dimensional space, introductory physics courses focus mostly on simpler problems where much is already known about an object's path. These problems are called *constrained motion problems*, and you'll be solving lots of them on the next two problem sets.

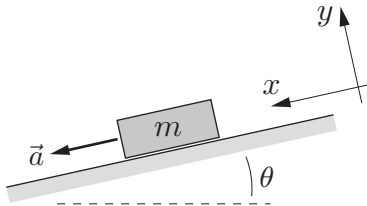
As a first example of a constrained motion problem, I'd like to return to the demonstration that I did during the first week of class: a low-friction cart gliding up and down an inclined track. Here's a simplified sketch of the situation, in which I've represented the cart as a simple block on an incline:



Suppose that we know the block's mass (say half a kilogram) and the angle of the incline (say 10 degrees). Our task is to predict the block's acceleration.

Please refer to your Constrained Motion Problem Worksheet, which outlines a method of solving problems like this. Notice that the worksheet follows the same general outline as our earlier worksheet for constant acceleration problems. However, there are several differences, as we'll see.

The first thing the worksheet asks us to do is to sketch the situation. I've already done that. But a plain sketch isn't enough. We now need to *label* the sketch with symbols and coordinate axes. So here's a labeled version of the sketch:



I first added the symbols m and θ to the sketch. Notice that I'm introducing algebraic symbols for these quantities, even though I know their numerical values. Please develop this same habit

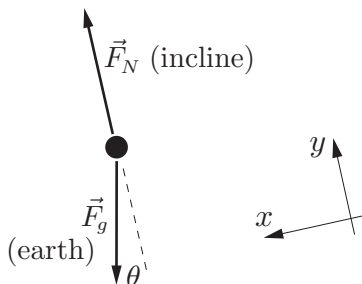
yourself! Next, I drew the acceleration vector on the sketch. We don't know the magnitude of this vector, but we do know (from observing the motion several weeks ago) that it points down-slope. Finally, I drew the x and y axes—in a funny way that seems very strange after you've worked too many projectile motion problems! Let me explain.

Nature never provides us with coordinate axes—we have to *invent* them. And we're free to choose the x and y directions to point any way we like, so long as they're perpendicular to each other. In projectile motion problems, it's always easiest to take one axis to be horizontal and the other vertical. But in constrained motion problems, it will often be more convenient to use axes that are tipped along a diagonal. You should choose whatever axes make the problem easiest to solve. And I've found, through years of experience, that it's always easiest if you choose one of the axes to point along the acceleration vector. Since the acceleration vector in this situation is tipped, I've drawn tipped axes. (If you're working a problem where the acceleration is zero, you might still choose tipped axes if this choice simplifies the analysis of the forces.)

One more point about the coordinate axes: In constrained motion problems you rarely need to keep track of where the origin is; you'll ordinarily use the axes only to keep track of which *direction* is x and which is y .

The next task on the worksheet is to list the known and unknown quantities. Some of the known quantities are $m = 0.5$ kg, $\theta = 10^\circ$, $a_y = 0$, and $g = 9.8$ N/kg. The unknown quantity of interest is a_x . The block's velocity is also unknown, but it's totally irrelevant in this problem: The block could be moving at any speed, in either direction—or it could be momentarily at rest, changing direction.

In the next section of the worksheet, our main task is to draw a *force diagram* for the object of interest—in this case, the block. Assuming that friction and air resistance are negligible, there are only *two* forces in this case: gravity and a normal (compression) force exerted by the incline. So here's the force diagram for the block:



Notice that the normal force on the block is *normal* (perpendicular) to the surface that exerts it, while the gravity force points straight down. I've also listed the agent of each force (to make sure I understand where it comes from), and I've again drawn my coordinate axes.

This section of the worksheet also asks us to describe the constraint. Our block is constrained to move along the incline, in a straight line. (The normal force somehow magically adjusts itself to ensure that this happens.) And the worksheet asks us to check that, according to our force diagram, the *net* (total) force points in the same direction as the acceleration. Looking at the diagram, I can see that the y components of the two forces appear to cancel out, but there's nothing to balance the x component of the gravity force, which does indeed point in the direction of \vec{a} , down-slope.

After all this preparation, we're now ready to start writing equations to solve the problem. And what equations might those be? Newton's second law, of course! This is the fundamental principle that you're going to apply: The vector sum of the forces determines the acceleration. And in nearly all constrained motion problems, you'll want to write this equation in component

form. Often it's nice to work with the two components in parallel, in two columns:

$$\sum F_x = ma_x \qquad \sum F_y = ma_y$$

We can immediately simplify the second equation by setting $a_y = 0$.

Now what? Well, look at the left-hand side of either equation. The \sum symbol means to *add* the indicated components of all the various forces acting on the object. This sum will contain one term for each force in your force diagram. We have just two forces, so our equations become:

$$F_{N,x} + F_{g,x} = ma_x \qquad F_{N,y} + F_{g,y} = 0$$

We now have four force components to deal with. The next step (in most constrained motion problems, including this one) is to relate these force components to the *magnitudes* and *angles* of the force vectors. Referring to the force diagram and using a bit of trig, we have:

$$\begin{aligned} F_{N,x} &= 0 \\ F_{N,y} &= |\vec{F}_N| \\ F_{g,x} &= |\vec{F}_g| \sin \theta \\ F_{g,y} &= -|\vec{F}_g| \cos \theta \end{aligned}$$

(Please be sure that you completely understand each of these relations. For \vec{F}_g , draw the triangle and use sohcahtoa.)

Plugging in these four formulas for the force components, the two components of Newton's second law become:

$$0 + |\vec{F}_g| \sin \theta = ma_x \qquad |\vec{F}_N| - |\vec{F}_g| \cos \theta = 0$$

From here on, every constrained motion problem is different. Usually it's a good idea to look back at your list of known and unknown quantities, add to this list if necessary, and then think about what you're trying to find out and what other information you might have. In this problem we can add $|\vec{F}_N|$ to our list of unknown quantities. But for $|\vec{F}_g|$ we can plug in the simple formula mg , to obtain:

$$mg \sin \theta = ma_x \qquad |\vec{F}_N| - mg \cos \theta = 0$$

Lo and behold, the m cancels in the first equation, and we're left with a formula for our desired answer:

$$a_x = g \sin \theta$$

At this point, you might be tempted to just plug in the numbers given and call it done. But first, it's a good idea to ask whether the *formula* for the answer makes sense. Is it reasonable that a_x should be proportional to the local gravitational constant? Yes. Is it reasonable that a_x should increase from zero when $\theta = 0$ to g when $\theta = 90^\circ$, as the sine function predicts? Absolutely. So you see, we've solved not just one problem, but a whole class of problems, for all possible values of g and θ . That's one of the many advantages of working algebraically, instead of plugging in numerical values from the start. (Other advantages: You don't have to waste time punching buttons on your calculator at every step, and you save even more arithmetic by canceling out the mass, which turns out to be totally irrelevant to the answer.)

If you *do* plug in $g = 9.8 \text{ m/s}^2$ and $\theta = 10^\circ$, you'll find that $a_x = 1.7 \text{ m/s}^2$. And then, as before, the worksheet asks: Is the sign correct? Yes, we expected a positive a_x . Are the units correct?

Yes, an acceleration should be in m/s^2 . And is the magnitude reasonable? Sure, we would expect an acceleration considerably less than g on an incline of only ten degrees.

At this point you may be wondering why I even bothered to write down the equation for the y component of Newton's second law, since we ended up not using it. The main reason is that in *most* problems you *will* need both components, so it's a good habit to write them both down from the start. But even in this problem, the y equation tells us something interesting—that the magnitude of the normal force is *less* than the object's weight (mg), by a factor of $\cos\theta$. At $\theta = 10^\circ$, this factor is 98.5% so the normal force is almost as strong as the object's weight. But at larger angles the normal force becomes much less, dropping to zero (as expected!) when $\theta = 90^\circ$.

That's all I can think of to say about a block sliding on a frictionless incline. But the real point of this example is to illustrate how to solve constrained motion problems in general. So please pause a moment to think about all the ways in which this problem could be modified. If there *is* friction, how would you take it into account? What if the incline were the curved surface of a hill or a valley? What if the block is being pulled by a string, strung over a pulley, connected to another hanging block? Although the details are different in each constrained motion problem, the same basic principles and methods (including Newton's second law) apply to them all.