Physics 2710 (Schroeder) Name ______________________________
fall 2012

Problem Set 5

(due Wednesday, October 10, 5:00 p.m.)

1. A definite-momentum wavefunction can be expressed by the formula $\psi(x) = A(\cos kx + i\sin kx)$, where A and k are constants.

   (a) How is the constant k related to the particle's momentum p_x? (Justify your answer.)

   (b) Show that, if a particle has such a wavefunction, you are equally likely to find it at *any* position x.

   (c) Explain why the constant A must be infinitesimal, if this formula is to be valid for all x.

   (d) Show that this wavefunction satisfies the differential equation $-i\hbar(d\psi/dx) = p_x\psi$.

2. (Gillespie, Exercise 7.) The complex function e^{ikx} (k real) is *defined* by $e^{ikx} \equiv \cos kx + i\sin kx$. Prove from this definition that e^{ikx} has the following properties:

   (a) $(e^{ikx})^* = e^{-ikx}$ (b) $e^{ik_1x} \cdot e^{ik_2x} = e^{i(k_1+k_2)x}$ (c) $|e^{ikx}|^2 = 1$ (d) $\frac{d}{dx}e^{ikx} = ike^{ikx}$

3. You can visualize a complex number by plotting it as a point in a plane, with the real part along the horizontal axis and the imaginary part along the vertical axis. On a single sheet of graph paper, plot and label each of the following complex numbers:

   (a) 3 (b) $3i$ (c) $3+3i$ (d) $-1.5+i$ (e) $e^{i\theta}$, for $\theta = 0$, $\pi/4$, $\pi/2$, π, and $3\pi/2$

   (f) $2e^{i\theta}$, for each of the θ values in part (e)

4. The formula for a "properly constructed" wavepacket is

$$\psi(x) = Ae^{ik_0x}e^{-ax^2},$$

   where A, a, and k_0 are constants.

   (a) Compute and sketch $|\psi(x)|^2$ for this wavefunction.

   (b) Show that the constant A must equal $(2a/\pi)^{1/4}$. (Hint: The probability of finding the particle somewhere between $x = -\infty$ and $x = \infty$ must equal 1. Use Mathematica to do the integral, or look it up in a table, but show exactly how you did it either way.)

   (c) The standard deviation Δx can be computed as $\sqrt{\overline{x^2} - \overline{x}^2}$, where the over-bar indicates an average. The average value of x is just the sum of all values of x, weighted by their probabilities:

$$\overline{x} = \int_{-\infty}^{\infty} x\,|\psi(x)|^2\,dx,$$

   and similarly for $\overline{x^2}$. Use these formulas to show that for this wavepacket, $\Delta x = 1/(2\sqrt{a})$.

   (d) The Fourier transform of a function $\psi(x)$ is defined as

$$\widetilde{\psi}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,\psi(x)\,dx.$$

   For *extra credit*, carry out this integral to show that $\widetilde{\psi}(k) = (A/\sqrt{2a})\exp[-(k-k_0)^2/4a]$ for our properly constructed wavepacket. Sketch $\widetilde{\psi}(k)$ in any case.

   (e) The function $\widetilde{\psi}(k)$ plays the same role for measurements of k that $\psi(x)$ plays for measurements of x. That is, the square modulus of this function, integrated over any range of k values, equals the probability of finding k within that range (if you were to measure k). What is the *most* likely k value?

(f) Using formulas analogous to those in part (c), show that, for this wavefunction, $\Delta k = \sqrt{a}$. (Hint: The standard deviation does not depend on k_0, so you can simplify the calculation by setting $k_0 = 0$ from the start.)

(g) Compute Δp_x for this wavefunction, and check whether the uncertainty principle is satisfied.

5. Sketch a wavefunction for which the product $(\Delta x)(\Delta p_x)$ is much greater than $\hbar/2$. Explain how you would estimate Δx and Δp_x for your wavefunction.

6. Make a rough estimate of the minimum energy of a proton confined inside a box of width 10^{-15} m (the size of an atomic nucleus).

7. For ultrarelativistic particles such as photons or high-energy electrons, the relation between energy and momentum is not $E = p^2/2m$ but rather $E = pc$. (This formula is valid for massless particles, and also for massive particles with $E \gg mc^2$.)

(a) Find a formula for the allowed energies of an ultrarelativistic particle confined to a one-dimensional box of length L.

(b) Estimate the minimum energy of an electron confined in a box of width 10^{-15} m. It was once thought that atomic nuclei might contain electrons; explain why this would be unlikely.

(c) A nucleon (proton or neutron) can be thought of as a bound state of three quarks that are approximately massless, held together by a very strong force that confines them inside a box of width 10^{-15} m. Estimate the minimum energy of three such particles (assuming all three to be in the ground state), and divide by c^2 to obtain an estimate of the nucleon mass.

8. A quantum-mechanical particle in a one-dimensional box is in the $n = 2$ definite-energy state. As discussed in class (and in your textbook), this means that its wavefunction oscillates in time according to the "wiggle factor" $e^{-i\omega t}$, where $\omega = E_2/\hbar$.

(a) Sketch the real and imaginary parts of this wavefunction for the following values of t: 0, $\pi/4\omega$, $\pi/2\omega$, π/ω, $3\pi/2\omega$, $2\pi/\omega$.

(b) Show that the probability distribution, $|\Psi|^2$, is independent of time, and sketch this function.

9. Not all wavefunctions are definite-energy wavefunctions! However, all wavefunctions *can* be expressed as linear superpositions of definite-energy wavefunctions. As a simple example, a particle in a rigid box could be in the following superposition of the $n = 1$ and $n = 2$ definite-energy wavefunctions:

$$\psi(x) = \frac{1}{\sqrt{2}}\big[\psi_1(x) + \psi_2(x)\big],$$

where ψ_1 and ψ_2 are the usual (normalized) definite-energy wavefunctions.

(a) Show that this wavefunction is properly normalized. (Hint: Definite-energy wavefunctions with distinct energy values are always "orthogonal," so the cross-terms integrate to zero.)

(b) Sketch this wavefunction.

(c) Sketch the square modulus of this wavefunction.

(d) If a particle has this wavefunction at time zero, we can find its wavefunction at later times by slipping the appropriate "wiggle factors" into each term:

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\big[\psi_1(x)e^{-i\omega_1 t} + \psi_2(x)e^{-i\omega_2 t}\big],$$

where $\omega_n = E_n/\hbar$ as usual. Compute the square modulus of this function, and express it in a way that doesn't involve i (since it must, after all, be a real-valued function). You should find that the time dependence does *not* cancel out: a factor of $\cos[(\omega_2 - \omega_1)t]$ remains. Sketch (or use a computer to plot) the function $|\Psi(x,t)|^2$ for a few different times, to show its behavior.