Electromagnetic Theory
Fall 2019

Problem Set 2

(due Friday, September 13, 4:00 pm)

1. **Constructing vector derivatives and integrals.** The accompanying worksheet shows a contour plot of a scalar function g in two dimensions. This function measures a quantity that I'll call *globbiness* (sorry), measured in units called *globs*. Each contour denotes a line of constant globbiness, successive contours differ by one glob, and the numerical value in globs is written on every fifth contour. To work this problem you will need a sharp pencil and a ruler that measures centimeters. A plastic triangle for constructing perpendiculars is helpful but not absolutely necessary.

   (a) Directly on the worksheet, carefully draw an arrow to represent the vector field $\mathbf{v} \equiv \nabla g$ at each of the twelve grid points labeled by lower-case letters. To do this you will need to measure the distance between nearby contour lines and do a short calculation for each location. It is necessary to choose a convention for the lengths of the arrows you draw; let the convention be that an arrow of length 1 cm represents a field strength of 1 glob/cm, with stronger and weaker field strengths represented by proportionally longer and shorter arrows.

   (b) Measure the x and y components of each of the arrows you constructed in part (a), and record their values in the table on the worksheet. (Try to estimate each component to the nearest half-millimeter on your ruler; there's no need to try to be more accurate than that.) Use the values in this table for all subsequent operations on $\mathbf{v}$.

   (c) Compute the line integral of $\mathbf{v} \cdot d\mathbf{l}$ along the path DBA, by breaking the path into four equal segments and approximating the value of $\mathbf{v}$ along each segment to be constant, equal to the value at the center of the segment (which is recorded in your table). Do the same for the line integral along path DCA. Also, directly from the contour plot, estimate the values $g(A)$ and $g(D)$. The fundamental theorem for gradients says that

   $$\int_D^A \mathbf{v} \cdot d\mathbf{l} = g(A) - g(D)$$

   (and therefore that this line integral is path independent). Do your computations verify the theorem? (To answer this last question you'll need to briefly discuss the amount of uncertainty in your measurements.)

   (d) In two dimensions, the curl of a vector function is a number, defined as

   $$\nabla \times \mathbf{v} \equiv \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}.$$

   Estimate $\nabla \times \mathbf{v}$ at each of the four interior points W, X, Y, and Z, in each case using the values of $\mathbf{v}$ at the four surrounding points to estimate the partial derivatives. For instance,

   $$\frac{\partial v_y(W)}{\partial x} \approx \frac{v_y(d) - v_y(c)}{\Delta},$$

where $\Delta = 4$ cm is the distance between c and d. Add up your four values for the curl and multiply by the appropriate area to obtain an estimate of the integral $\int(\nabla\times\mathbf{v})da$ over the entire square area shown. Also estimate the circulation $\oint \mathbf{v}\cdot d\mathbf{l}$ for the closed path ACDB. The fundamental theorem for curls states that

$$\int(\nabla\times\mathbf{v})da = \oint \mathbf{v}\cdot d\mathbf{l}.$$

Do your calculations verify this theorem? (They should, *exactly.*) Write a short paragraph explaining why this "fundamental" theorem is in fact trivial.

(e) Estimate $\nabla\cdot\mathbf{v}$ at each of the interior points W, X, Y, and Z, using a method analogous to the way you calculated curls in part (d). Combine your results to find $\int(\nabla\cdot\mathbf{v})d\tau$ for the two-dimensional "volume" (really an area) enclosed by the entire square. Also estimate the flux $\oint \mathbf{v}\cdot d\mathbf{a}$ for the closed "surface" (really a line) bounding the square, breaking the surface into eight equal segments and approximating $\mathbf{v}$ along each segment by its value at the center (a grid point). The fundamental theorem for divergences states that

$$\int(\nabla\cdot\mathbf{v})d\tau = \oint \mathbf{v}\cdot d\mathbf{a}.$$

Do your calculations verify this theorem? (They should, *exactly.*) Briefly explain why this "fundamental" theorem is in fact trivial.

2. Prove that the divergence of a curl is always zero. Then check this theorem for the function $x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$.

3. Starting with a clear picture or two, derive the formulas for $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ in terms of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$. (The pictures here are important! If you're not sure how to visualize these vectors, ask for a hint.)

4. (a) Check the divergence theorem for the function $r^2\hat{\mathbf{r}}$, using as your volume a sphere of radius R, centered at the origin. (Compute the divergence from scratch, using rectangular coordinates, but do the integrals in spherical coordinates.) (b) Do the same for the function $(1/r^2)\hat{\mathbf{r}}$. (In this case we'll compute the divergence in class, and you may simply quote that result.)

5. Consider the three-dimensional function $\rho(r) = k\delta(r-a)$, where k and a are positive constants and r is the distance from the origin. Note that the delta function is a *one*-dimensional function of r only. (a) *Describe* this function. How would you visualize it? (b) Calculate the three-dimensional integral $\int \rho(r)d\tau$ over all space, using spherical coordinates. (c) If $\rho(r)$ represents the mass density (mass per unit volume) of an object whose total mass is m, what is the value of k?

6. For "Theorem 1" in Griffiths's section on the Theory of Vector Fields, show that (d) $\Rightarrow$ (a), (a) $\Rightarrow$ (c), (c) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), and (c) $\Rightarrow$ (a).