Electromagnetic Theory
Fall 2019

Problem Set 11

(due Wednesday, December 4, 12:30 pm)

1. Calculate the energy stored in a long cylindrical solenoid with length ℓ, radius R, current I, and n turns per unit length, in three different ways: (a) using the formula $\frac{1}{2}LI^2$ for the energy stored in an inductor; (b) using the formula $\frac{1}{2}\oint(\mathbf{A}\cdot\mathbf{I})dl$ in terms of the current and the vector potential; and (c) using the formula $(1/2\mu_0)\int B^2 d\tau$ in terms of the magnetic field. You may look up the formulas for L, $\mathbf{A}$, and $\mathbf{B}$ in your textbook and/or class notes.

2. Consider the possibility of storing energy in the magnetic field of a superconducting solenoidal coil. About how large would such a coil have to be, to store enough energy to power a typical U.S. home for one night? Would it fit in your basement? Feel free to use any reasonable estimates for the total energy that needs to be stored and for the maximum magnetic field that can be attained, but state your estimates clearly and explain how you made them. Feel free to look up some numbers to test whether your estimates are reasonable.

3. A fat cylindrical wire, radius a, carries a constant current I, uniformly distributed over its cross section. A narrow gap in the wire, of width $w \ll a$, forms a parallel-plate capacitor, as the surfaces on each side of the gap become charged due to the current flow. Find the electric and magnetic fields in the gap, as functions of time and of the distance s away from the axis. (Assume that the charge on the surfaces is zero at $t = 0$.) Sketch the magnetic field in the $s\phi$ plane, as viewed from the direction of the positively charged surface (so the current points into the page).

4. Consider the charging "capacitor" of Problem 3, above.

   (a) Find the energy density u_{em} and the Poynting vector $\mathbf{S}$ in the gap. Note especially the *direction* of $\mathbf{S}$. Check that the continuity equation for energy is satisfied.

   (b) Determine the total energy in the gap as a function of time. Calculate the total power flowing into the gap, by integrating the Poynting vector over the appropriate surface. Check that the power input is equal to the rate of increase of energy in the gap, that is, that Poynting's theorem is satisfied. (In this case $W = 0$, because there is no charge in the gap.)

5. Consider the electric and magnetic fields

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\theta(r - vt)\hat{\mathbf{r}}, \qquad \mathbf{B}(\mathbf{r},t) = 0,$$

   where $\theta(x)$ is the step function, equal to zero when its argument is negative and 1 when its argument is positive. Show that these fields satisfy all four of Maxwell's equations, and determine ρ and $\mathbf{J}$. Describe the physical situation that gives rise to these fields.

6. Find the fields, and the charge and current distributions, corresponding to the potentials

$$V(\mathbf{r},t) = 0, \qquad \mathbf{A}(\mathbf{r},t) = -\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\hat{\mathbf{r}}.$$

Then use the gauge function $\lambda = -(1/4\pi\epsilon_0)(qt/r)$ to transform these potentials, and comment on the result.

7. For your final writing exercise this semester, please write up a formal presentation of *one* of the following (your choice!):

- Problem Set 10, Problem 4, and its solution. This problem concerns a simple electromagnetic generator. Be sure to include enough diagrams to make the solution clear. Also please include at least one numerical example and discuss the numbers briefly.

- Problem Set 10, Problem 5, and its solution. This problem concerns the electric field around a solenoidal electromagnet with a time-dependent current. Be sure to include the sketches of the scenario and of your results. Also please include a numerical example and discuss the numbers briefly.

- Problem Set 11, Problems 1 and 2, and their solutions. These problems concern the energy associated with a solenoidal electromagnet. Please include an illustration of a solenoid and its magnetic field.

- Problem Set 11, Problems 3 and 4, and their solutions. These problems concern the electric field, displacement current, and resulting magnetic field associated with a narrow gap in a fat current-carrying wire. Be sure to include sketches of the scenario and of the fields. Also please work out a numerical example and discuss the numbers briefly.

- Explain how the idea that "the boundary of a boundary is zero" is related to conservation of electric charge by way of Maxwell's equations. This is a topic that I will discuss in class; it may especially interest you if you enjoy abstract mathematics.

- Another topic of your choice, with permission. If you would like to write up a formal presentation of some other problem, or of some other topic in electromagnetic theory, then please describe it to me and I will let you know whether it is suitable for this assignment.

Regardless of your choice, please put extra effort into writing an introductory paragraph to motivate your topic. This paragraph should explain why the topic or problem is interesting and provide any other important context. Elsewhere, try to add insight and interpretation where appropriate, rather than presenting only a dry mathematical calculation. Use your best English spelling, punctuation, grammar, word choice, and so on. Present mathematical equations with correct punctuation, incorporating each equation into a grammatically correct sentence. Where illustrations are needed, you may either draw them neatly by hand or use a computer drawing program.

As always, the target audience for this assignment is a physics student who is taking the same course but who has not yet worked the problem or seen the specific ideas that you're writing about. Try to present everything at an appropriate level of detail for such a student.

The target length of this writing assignment is about three double-spaced pages. If you find yourself writing a great deal more than that, then please think carefully about which details you can reasonably omit.