Physics 4610
Quantum Mechanics
Exam 1
Spring Semester 2014

Notes:
You may use your textbook and a table of integrals.

Name: Key
1. A particle, which is confined to an infinite square well of width L, has a wavefunction given by,

\[ \psi(x) = \frac{2}{\sqrt{L}} \sin\left(\frac{2\pi}{L} x\right) \]

a) Calculate the expectation value of position x and momentum p.

\[ \langle x \rangle = \frac{2}{L} \int_0^L x \left( \frac{2\pi}{L} x \right) \, dx = \frac{L}{2} \]

\[ \langle p \rangle = \frac{2}{L} \int_0^L \frac{d}{dx} \left[ \sin\left(\frac{2\pi}{L} x\right) \right] \, dx = \ldots \]

\[ \langle p \rangle = 0 \]

b) Calculate the expectation of energy E.

\[ \langle E \rangle = \int \psi^*(x) H \psi(x) \, dx \]

In this case \( \psi(x) = \psi_2(x) \) of infinite square well,

\[ H \psi_2(x) = E_2 \psi_2(x) \Rightarrow \langle E \rangle = E_2 \int \psi_2^* \psi_2 \, dx = \frac{4\pi^2 \hbar^2}{2mL^2} \]

\[ \langle E \rangle = E_2 \]

c) Calculate the uncertainty \( \sigma_E \) and explain your results.

\[ \langle E^2 \rangle = \int \psi^*(x) \left( \frac{H^2}{E_2} \psi_2(x) \right) \, dx = \frac{E_2^2}{E_2} \]

\[ \sigma_E^2 = \langle E^2 \rangle - \langle E \rangle^2 = E_2^2 - E_2^2 = 0 \]

The particle is in an stationary state (well defined energy states), thus \( \Delta E = 0 \).
2. The state of a particle confined to an infinite square well of width $L$ is given as

$$\psi(x, 0) = N x$$

where $N$ is a normalization constant.

a) Normalize the wave function $\psi(x, 0)$.
b) Express this wavefunction as a superposition of the eigenstates of the infinite square well.
c) What is the wavefunction of particle at a later time $t$?
d) What is the probability of finding the particle with the energy $E = E_2$?

\[
\int [4(x_{10})]^2 dx = 1 \Rightarrow N^2 \int_0^L x^2 dx = 1
\]

\[
N^2 \left[ \frac{L^3}{3} \right] = 1 \Rightarrow N = \frac{\sqrt{\frac{3}{L}}}{\frac{1}{L}} = \frac{\sqrt{3}}{\frac{1}{L}}
\]

\[
\psi(x_{10}) = \frac{1}{L} \sqrt{\frac{3}{L}} x
\]

\[
b(x) = \sum_n c_n \psi_n(x) \Rightarrow c_n = \int \psi(x_{10}) \psi_n(x) dx
\]

but $\psi_n(x) = \frac{2}{L} \sin \frac{n\pi}{L} x$ $\Rightarrow c_n = \frac{\sqrt{6}}{L^2} \int_0^L \sin \frac{n\pi}{L} x dx$

\[
c_n = \frac{\sqrt{6}}{n\pi} (-1)^{n+1} \Rightarrow
\]

\[
\psi(x_{10}) = \sum_n \frac{\sqrt{6}}{n\pi} (-1)^{n+1} \cdot \frac{2}{L} \sin \frac{n\pi}{L} x
\]

\[
P(E = E_2) = |c_2|^2 = \left| \frac{\sqrt{6}}{2\pi} \right|^2 = \frac{6}{4\pi^2} = 0.15
\]

\[
= 15\%
\]
3. An electron is acted by a potential $V(x)$ within the region $0 \leq x \leq a$. Its wavefunction is given as

$$
\psi(x) = N \sin \left( \frac{\pi x}{a} \right) e^{-i\omega t} \quad 0 \leq x \leq a \\
\psi(x) = 0 \quad \text{otherwise}
$$

(a) Calculate the normalization constant $N$.
(b) Calculate the probability of finding the electron in the interval $a/4 \leq x \leq 3a/4$.
(c) Find the potential $V(x)$.

\[ a) \quad \int_{a/4}^{3a/4} \left| \psi(x) \right|^2 dx = 1 \Rightarrow \int_{0}^{a} \frac{N^2 \sin^2 \left( \frac{\pi x}{a} \right)}{a} dx = 1 \]

\[ \Rightarrow N = \sqrt{\frac{2}{a}} \]

\[ \psi(x,t) = \sqrt{\frac{2}{a}} \sin \frac{\pi x}{a} \times e^{-i\omega t} \]

\[ b) \quad P = \int_{a/4}^{3a/4} \left| \psi(x,t) \right|^2 dx = \frac{9}{a} \int_{0}^{a} \frac{N^2 \sin^2 \left( \frac{\pi x}{a} \right)}{a} dx \]

\[ = \frac{1}{2} + \frac{1}{6} = 0.83 \]

\[ c) \quad -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x,t) + \nabla^2 \psi(x,t) = i\hbar \frac{\partial}{\partial t} \psi(x,t) \]

\[ \Rightarrow \nabla = -\frac{\hbar^2 \pi^2}{2ma^2} + i\omega \]

\[ \Rightarrow \nabla = -\frac{\hbar^2 \pi^2}{2ma^2} + i\omega \]
4. At \( t = 0 \), a particle in a harmonic-oscillator potential is in the initial state

\[
\psi(x, 0) = \frac{1}{\sqrt{5}} \psi_1(x) + \frac{2}{\sqrt{5}} \psi_2(x)
\]

a) Calculate the expectation value of energy in the state \( \psi(x, 0) \).

b) Find the state of the particle \( \psi(x, t) \) at a later time \( t \). Is this state a stationary state?

c) Calculate the expectation value of \( x \) for the state \( \psi(x, t) \).

d) What is the frequency of oscillation of this expectation value?

\[
\langle E \rangle = \int \left[ \frac{1}{2} \psi_1^*(x) \psi_1(x) + \frac{2}{5} \psi_2^*(x) \psi_2(x) \right] dx
\]

\[
= \int \left[ \frac{1}{5} E_1 \psi_1(x) \psi_1(x) + \frac{2}{5} E_2 \psi_2(x) \psi_2(x) \right] dx
\]

\[
\langle E \rangle = \frac{1}{5} E_1 + \frac{4}{5} E_2 = \frac{1}{5} \left( \frac{E_1 + E_2}{2} \right) + \frac{4}{5} \left( \frac{E_1}{2} \right) = \frac{23}{10} \hbar
\]

b) \( \psi(x, t) = \frac{1}{\sqrt{5}} \psi_1(x) e^{-i E_1 \frac{t}{\hbar}} + \frac{2}{\sqrt{5}} \psi_2(x) e^{-i E_2 \frac{t}{\hbar}} \)

This state is NOT a stationary state because its prob. density depends on time.

c) \( \langle x \rangle = \int \left( \frac{1}{\sqrt{5}} \psi_1(x) e^{-i E_1 \frac{t}{\hbar}} + \frac{2}{\sqrt{5}} \psi_2(x) e^{-i E_2 \frac{t}{\hbar}} \right) x \left( \text{Same thing} \right) dx
\]

\[
\langle x \rangle = \int \left( \frac{1}{\sqrt{5}} \psi_1(x) + \frac{2}{\sqrt{5}} \psi_2(x) \right) x \left( \text{Same thing} \right) dx
\]

\[
= \frac{1}{5} \int x \psi_1(x) dx + \frac{4}{5} \int x \psi_2(x) dx + \frac{2}{5} \int x \psi_2(x) \cdot 2 \cos \left( \frac{E_1 - E_2}{\hbar} \right) t dx
\]

\[
\langle x \rangle = \frac{2}{5} \cdot 2 \cdot \int \frac{1}{m \omega} \cos \left( \frac{E_1 - E_2}{\hbar} \right) t dx
\]

\[
\omega = \text{frequency} = \frac{E_2 - E_1}{\hbar}
\]