Physics 3500
Analytical Mechanics
Exam 2
Fall Semester 2013

Notes:

- The test is open book.
- Time limit = 75 minutes
- Show all your work

NAME:

Key

1. A particle's pontential energy is $V = k(x^2 + y^2 + z^2)$, where k is a constant.

a) What is the force acting on the particle?

b) Prove that this force is conservative.

a) $\vec{F} = -\vec{\nabla} V = -\frac{\partial V}{\partial x}\hat{i} - \frac{\partial V}{\partial y}\hat{j} - \frac{\partial V}{\partial z}\hat{k}$

$$\boxed{\vec{F} = -2k(x\hat{i} + y\hat{j} + z\hat{k})}$$

b) $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2kx & -2ky & -2kz \end{vmatrix}$

$= \hat{i}[0-0] + \hat{j}[0-0] + \hat{k}[0-0] = 0$

Thus $\vec{F}$ is a conservative force.

2. The first Russian spacecraft Sputnik-I had a perigee of 227 km above the earth surface. At perigee its speed was 8 km/s. Take radius of the earth $R_e = 6400 km$.

a) Calculate its apogee, speed at the apogee.

$$r_p = 227 + 6400\ km = 6627\ km = 6.627 \times 10^6\ m$$

$$E = \frac{1}{2}\cancel{m}v_p^2 - \frac{G\cancel{m}M}{r_p} = -\frac{G\cancel{m}M}{2a}$$

Solve for a

$v_p \to 8000\ m/s$, $r_p \to 6.627 \times 10^6\ m$

$$\Rightarrow \boxed{a = 7.025 \times 10^6\ m}$$

$$r_p = a(1-\epsilon) \to \epsilon = 1 - r_p/a = 0.057$$

$$r_a = a(1+\epsilon) = 7.025 \times 10^6 (1 + 0.057) = \boxed{7.423 \times 10^6\ m}$$

$$r_a v_a = r_p v_p \to v_a = v_p\, r_p/r_a = \boxed{7.14\ km/s}$$

b) Suppose we want the spacecraft to escape the gravitational field of the earth. When the spacecraft is at its apogee, an impulse velocity change is applied just sufficient to place it into the escape trajectory. Calculate the magnitude of the velocity change.

$$E_{min} = \frac{1}{2}mv^2 - \frac{GmM}{r} = 0$$

$$v = \sqrt{\frac{2GM}{r = r_a}} = \sqrt{\frac{2 \times 4.014 \times 10^{14}}{7.423 \times 10^6}} = 10.4\ km/s$$

$$\Delta v = 10.4 - 7.14 = \boxed{3.26\ km/s}$$

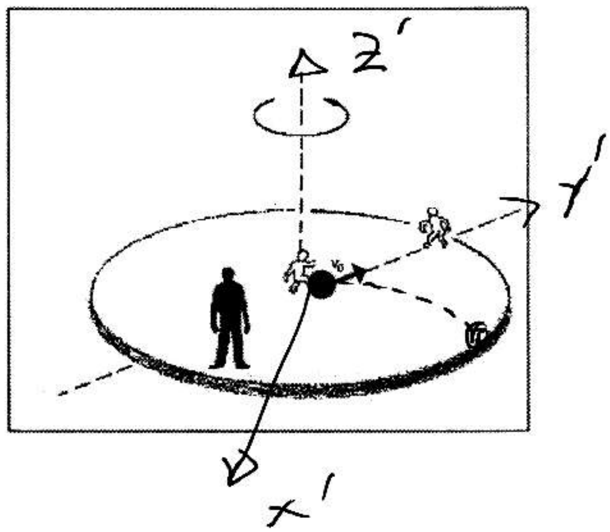
3. Two boys are standing on a merry-go-round of radius 3 m with an angular speed of ω = 6 rev per **minute**. The boy at the center rolls a $0.5-kg$ basketball to the other boy with an initial speed of $v_0 = 1.5\ m/s$. This problem is studied from point of view of the observer on the merry-go-round. Be sure to convert ω to rad/s.

Calculate the Coriolis force acting on the ball as it is released. Use your result to calculate the Coriolis acceleration due to this force. Make a rough sketch of the path of the ball on the figure.

a) $\vec{V}_0 = 1.5\,\hat{j}'$

$$\omega = 6\ \text{rev}/\text{min} = \frac{6}{60}(2\pi)$$

$$\omega = 0.628\ \text{rad}/s$$



$$\vec{F}_c = -2m\vec{\omega} \times \vec{v}'$$

$$= -2m(\omega\hat{k}') \times (V_0\hat{j}')$$

$$\boxed{\vec{F}_c = +2m\omega V_0\,\hat{i}'} = 2(0.5)(0.628)(1.5)\hat{i}'$$

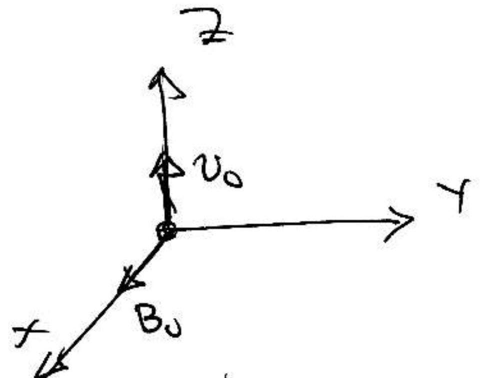
$$\boxed{\vec{F}' = 0.942\,\hat{i}'}\ N$$

$$a_c = |\vec{F}_c'|/m = \frac{2\cancel{m}\omega v_0}{\cancel{m}} = \boxed{2\omega V_0}$$

$$\boxed{a_c = 1.884\ m/s^2}$$

4. A charged particle of mass m and positive charge q, enters in a region where there is a magnetic field of magnitude B_0 in the x-direction. At time $t=0$, the particle is at the origin, and its initial velocity is v_0 and is in the z-direction.

a) Set up the differentail equations (**without solving them**) of motion for all three components x, y, and z. Use the definition $\omega = \dfrac{qB_0}{m}$.



$$\vec{F} = q\vec{v} \times \vec{B}$$

$$= q \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dot{x} & \dot{y} & \dot{z} \\ B_0 & 0 & 0 \end{vmatrix}$$

$$\boxed{\vec{F} = 0\hat{i} + qB_0\dot{z}\,\hat{j} - qB_0\dot{y}\,\hat{k}} = m\ddot{\vec{r}}$$

$$\Rightarrow \begin{cases} \ddot{x} = 0 \\ \ddot{y} = w\dot{z} \\ \ddot{z} = -w\dot{y} \end{cases} \quad \text{where } w = \frac{qB_0}{m}$$

b) Use the results of part(a) to solve the differetial equation (**only**) for z-component and derive z(t). Be sure to apply initial conditions and solve for the constants.

$\ddot{z} = -w\dot{y}$ from 2nd equat: $\ddot{y} = w\dot{z} \Rightarrow$

$$\dot{y} = wz + C$$

$$t=0 \quad 0 = 0 + C \Rightarrow C = 0$$

$$\boxed{\dot{y} = wz}$$

$$\ddot{z} = -w(wz)$$

$$\ddot{z} + w^2 z = 0$$

$$z = A\cos wt + B\sin wt$$

$t=0 \quad z=0 \Rightarrow \boxed{A=0} \rightarrow z = B\sin wt \Rightarrow \dot{z} = Bw\cos wt$

$t=0 \quad \dot{z} = V_0 = Bw \rightarrow B = \frac{V_0}{w}$

$z = V_0/w \sin wt$