Physics 3500
Analytical Mechanics
Exam 1
Fall Semester 2013

Notes:

- The test is open book.
- Time limit = 1 hour
- **Show all your work**

Key

1. Consider vectors $\vec{A}$ and $\vec{B}$ given by,

$$\vec{A} = 3\hat{i} + 4\hat{j} + c\hat{k}$$
$$\vec{B} = -2\hat{i} + 4\hat{j} + 5\hat{k}$$

where c is parameter to be determined later.

a) Calculate $\vec{N} = \vec{A} \times \vec{B}$

b) Determine the parameter c such that the vector $\vec{A}$ be perpendicular to the vector $\vec{B}$.

a) $$\vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & c \\ -2 & 4 & 5 \end{vmatrix} = \hat{i}(20-4c) + \hat{j}(-2c-15) + \hat{k}(12+8)$$

$$\boxed{\vec{N} = \hat{i}(20-4c) - \hat{j}(2c+15) + 20\hat{k}}$$

b) We need $\vec{A} \cdot \vec{B} = 0$

$$\downarrow$$

$$A_x B_x + A_y B_y + A_z C_z = 0$$

$$\downarrow$$

$$(3)(-2) + (4)(4) + (c)(5) = 0$$

$$-6 + 16 + 5c = 0 \rightarrow \boxed{c = -2}$$

2. The motion of a particle in a plane is given by the polar coordinates:

$$\vec{r} = 0.5(1-\cos\theta)\hat{e}_r \qquad (in\ meters)$$
$$\theta = 2t+1 \qquad (in\ radians)$$

where t is the time in seconds.

a) Calculate the velocity vector $\vec{v}$.

b) Calculate the magnitude of the velocity $|\vec{v}|$.

c) Find the angle between $\vec{r}$ and $\vec{v}$ at the time $t = 0.5$ second.

a) $\dot{\vec{r}} = 0.5(\sin\theta)\dot{\theta}\,\hat{e}_r + 0.5(1-\cos\theta)\dot{\theta}\,\hat{e}_\theta$

using $\dot{\theta} = \frac{d}{dt}(2t+1) = 2$

$\Rightarrow \boxed{\dot{\vec{r}} = \vec{V} = \sin\theta\,\hat{e}_r + (1-\cos\theta)\,\hat{e}_\theta}$

b) $|\vec{V}| = \sqrt{\sin^2\theta + (1-\cos\theta)^2} = \sqrt{2-2\cos\theta} = 2\sin\theta/2$

c) $\vec{r}\Big|_{t=0.5} = 0.5\left(1-\cos(2\text{ radians})\right)\hat{e}_r = 0.71\,\hat{e}_r$

$\vec{V}\Big|_{t=0.5} = \sin(2\text{ radians})\,\hat{e}_r + \left(1-\cos(2\text{ radians})\right)\hat{e}_\theta$

$\Rightarrow \vec{V}\Big|_{t=0.5} = 0.9\,\hat{e}_r + 1.42\,\hat{e}_\theta$

$\cos\alpha = \frac{\vec{r}\cdot\vec{V}}{|\vec{r}|\,|\vec{V}|} = \frac{(0.71)(0.9)}{(0.71)(1.68)} = 0.53 \Rightarrow \boxed{\alpha = 57.6^\circ}$

3. A ball of mass $m = 0.75kg$ radius $r = 15cm$ is dropped vertically at rest from the height of 12 m above the ground. An air resistance force in the form of $F(v) = c_2 v^2$ acts on the ball. (For this problem, you do not need to solve the equation of motion).

Calculate the speed of the ball as it reaches the ground. Compare your answer with the terminal speed of the ball (i.e. their percent difference).

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a) $mg = 0.75 \times 9.8 = 7.35\,N$

$c_2 = 0.22(0.3)^2 = 0.0198$

$v_t = \sqrt{\frac{mg}{c_2}} = 19.3\ m/s$ and $\tau = \frac{v_t}{g} = 1.97\ s$

$\Rightarrow v^2 = v_t^2\left(1 - e^{-2gy/v_t^2}\right) = 19.3^2\left[1 - e^{-\frac{2 \times 9.8 \times 12}{19.3^2}}\right]$

$v^2 = 19.3^2(1 - 0.53) \Rightarrow \boxed{v = 13.2\ m/s}$

Thus, the speed $v = 19.3$ compared to speed $v_t = 13.2$ has not reached the terminal speed yet. It is:

$\%\ diff. = 100 \times \frac{19.3 - 13.2}{13.2} = 46.2\ \%$

4. When a mass of $m = 0.2kg$ is suspended vertically from a spring of spring constant k, the spring expands by 15 cm. The mass and spring are at rest at this time. Call this point $x = 0$ with $x-$ axis directed vertically upward.

a) Calculate the spring constant k.
b) Suppose the mass is pulled down by 10 cm and released. Calculate the angular frequency ω and period T of this motion. Be sure to include the units.
c) Write the equations for the position $x(t)$ and the velocity $v(t)$ assuming that the mass was pulled down (by 10 cm) and released at $t = 0$. What is the velocity at the time $t = 2$?

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a)

$F_{net} = 0 \Rightarrow mg - kx = 0 \Rightarrow$

$k = \frac{mg}{x} = \frac{(0.2)(9.8)}{0.15}$

$\boxed{k = 13\ N/m}$

b) $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{13}{0.2}} = 8\ rad/s$

$T = \frac{2\pi}{\omega} = \frac{2(3.14)}{8} = 0.8\ s$

c) $x(t) = A\sin(\omega t + \varphi) \Rightarrow x(t) = -10\sin(8t + \varphi)$

at $t = 0$ $x(t=0) = -10 = -10\sin(0 + \varphi) \rightarrow \boxed{\varphi = \frac{\pi}{2}}$

$x(t) = -10\sin(8t + \frac{\pi}{2}) = -10\cos(8t)$

$V(t) = +80\sin 8t$

c) At $t = 2\ s \Rightarrow$
$V = 80\sin(16) = -23\ cm/s$